Course: B. Tech.
Semester II
Mathematics II (KAS 203)
Module 4: Complex Variable – Differentiation

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1.1 Introduction— Functions of a complex variable provide us some powerful and widely useful tools in mathematical analysis as well as in theoretical physics.

1.2 Applications—

1. Some important physical quantities are complex variables (the wave-function Ψ, a.c. impedance $Z(\omega)$).
2. Evaluating definite integrals.
3. Obtaining asymptotic solutions of differentials equations.
4. Integral transforms.
5. Many Physical quantities that were originally real become complex as simple theory is made more general. The energy $E_n \to E_n^0 + i\Gamma$ ($\Gamma^{-1} \to$ the finite life time).

1.3 Complex Algebra—A complex number $z(x, y) = x + iy$. Where $i = \sqrt{-1}$.

x is called the real part, labeled by **Re z**

y is called the imaginary part, labeled by **Im z**

Different forms of complex number—

1. Cartesian form— $z(x, y) = x + iy$
2. Polar form— $z(r, \theta) = r(\cos\theta + i\sin\theta)$
3. Euler form— $z(r, \theta) = re^{i\theta}$

Here, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(\frac{y}{x})$

1.4 Derivative of the function of complex variable—

The derivative of $f(z)$ is defined by—

$$f'(z) = \lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

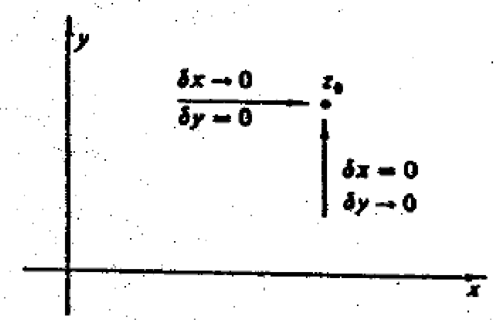
If $z = x + iy$ and $f(z) = u + iv$,

let consider that—

$$\delta z = \delta x + i\delta y\,,$$

$$\delta f = \delta u + i\delta v\,,$$

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$



Assuming the partial derivatives exist. Let us take limit by the two different approaches as in the figure. First, with δy = 0, we let δx→0,

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

For a second approach, we set δx = 0 and then let δy→ 0. This leads to –

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(\frac{\delta u}{\delta y} + i\frac{\delta v}{\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

1.5 Analytic function— A single valued function that possesses a unique derivative with respect to z at all points on a region R is called an Analytic function of z in that region.

Necessary and sufficient condition to be a function Analytic— A function $f(z) = u + iv$ is called an analytic function if—

1. $f(z) = u + iv$, such that u and v are real single valued functions of x and y and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in R.
2. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

1.6 Cauchy-Riemann equations— If a function $f(z) = u + iv$ is analytic in region R, then it must satisfies the relation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These equations are called as **Cauchy-Riemann equation.**

Cauchy-Riemann equation in polar form—

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \text{ and } -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

1.7 Laplace's equation— An equation in two variables x and y given by $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ is called as Laplace's equation in two variables.

1.8 Harmonic functions— If $f(z) = u + iv$ be an analytic function in some region R, then Cauchy-Riemann equations are satisfied. That implies—

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \ldots (1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \ldots (2)$$

Now, on differentiating (1) with respect to x and (2) with respect to y—

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x\, \partial y} \qquad \ldots (3)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial y \partial x} \qquad \ldots (4)$$

Assuming that $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, adding equations (1) and (2)—

$$\boldsymbol{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly, on differentiating (1) with respect to y and (2) with respect to x and subtracting them—

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

both u and v satisfy the Laplace's equation in two variables, therefore $f(z)=u+iv$ is called as **Harmonic function.** This theory is known as **Potential theory.**

If $f(z)=u+iv$ is an analytic function in which $u(x,y)$ is harmonic, then $v(x,y)$ is called as **Harmonic conjugate** of $u(x,y)$.

1.9 Orthogonal system— Consider the two families of curves $u(x,y)=c_1 \ldots$ (1) and $v(x,y)=c_2 \ldots$ (2).

Differentiating (1) with respect to x –

$$\frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial x}}=m_1 \qquad \text{[by using Cauchy's Riemann equations for analytic function]}$$

Differentiating (2) with respect to x –

$$\frac{dy}{dx}=-\frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial x}}=m_2$$

Here, $m_1 m_2=-1$ therefore every **analytic function** $f(z)=u(x,y)+iv(x,y)$ represents two families of curves $u(x,y)=c_1$ and $v(x,y)=c_2$ form an **orthogonal system**.

1.10 Determination of Analytic function whose real or imaginary part is known—

If the real or the imaginary part of any analytic function is given, other part can be determined by using the following methods—

(a) Direct Method

(b) Milne-Thomson's Method

(c) Exact Differential equation method

(d) Shortcut Method

1.10.1 Direct Method—

Let $f(z) = u + iv$ is an analytic function. If v is given, then we can find u in following steps–

Step-I Find $\frac{\partial u}{\partial x}$ by using C-R equation $\left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right]$

Step-II Integrating $\frac{\partial u}{\partial x}$ with respect to x to find u with taking integrating constant $f(y)$

Step-III Differentiate u (from step-II) with respect to y. $\left[\text{Evaluate } \frac{\partial u}{\partial y} \text{ containing } f'(y)\right]$

Step-IV Find $\frac{\partial u}{\partial y}$ by using C-R equation $\left[\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$

Step-V by comparing the result of $\frac{\partial u}{\partial y}$ from step-III and step-IV, evaluate $f'(y)$

Step-VI Integrate $f'(y)$ and evaluate $f(y)$

Step-VII Substitute the value of $f(y)$ in step-II and evaluate u.

Example— If $f(z) = u + iv$ represents the analytic complex potential for an electric field and $v = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function u.

Solution— Given that –

$$v = x^2 - y^2 + \frac{x}{x^2+y^2}$$

Hence—

$$\frac{\partial v}{\partial x} = 2x + \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}$$

$$= 2x + \frac{y^2-x^2}{(x^2+y^2)^2} \quad \dots (1)$$

Again,

$$\frac{\partial v}{\partial y} = -2y + \frac{-2xy}{(x^2+y^2)^2} \quad \dots (2)$$

So, u and v must satisfy the Cauchy's-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots (3) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots (4)$$

Now, from equations (2) and (3)—

$$\frac{\partial u}{\partial x} = -2y + \frac{-2xy}{(x^2+y^2)^2}$$

On integrating with respect to x—

$$u = -2y \int dx - y \int \frac{2x}{(x^2+y^2)^2} dx$$

$$u = -2xy + \frac{y}{x^2+y^2} + f(y)$$

Differentiating with respect to y—

$$\frac{\partial u}{\partial y} = -2x - \frac{y^2-x^2}{(x^2+y^2)^2} + f'(y) \qquad \dots (5)$$

Again, from equations (1) and (4)—

$$\frac{\partial u}{\partial y} = -2x - \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \dots (6)$$

On comparing (5) and (6)—

$f'(y) = 0$, so $f(y) = k$

Hence, $u = -2xy + \frac{y}{x^2+y^2} + k$ ans

1.10.2 Milne-Thomson's Method—

Let a complex variable function is given by $f(z) = u(x, y) + iv(x, y)$ ... (1)

Where— $z = x + iy$ and $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$.

Now, on writing $f(z) = u(x, y) + iv(x, y)$ in terms of z and $\bar{z}$ –

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$$

Considering this as a formal identity in the two variables z and $\bar{z}$, and substituting $z = \bar{z}$ –

$$\therefore f(z) = u(z, 0) + iv(z, 0) \quad \ldots (2)$$

Equation (2) is same as the equation (1), if we replace x by z and y by 0.

Therefore, to express any function in terms of z, replace x by z and y by 0. This is an elegant method of finding $f(z)$, 'when its real part or imaginary part is given' and this method is known as **Milne-Thomson's Method.**

Example— If $f(z) = u + iv$ represents the complex potential for an electric field and $v = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function u. (By using Milne-Thomson's method)

Solution— Given that— $f(z) = u + iv$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \qquad \text{[by using cauchy − Riemann equation]}$$

Hence, $\therefore \frac{\partial f}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = \left(-2y + \frac{-2xy}{(x^2+y^2)^2}\right) + i\left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)$

by using Milne-Thomson's method (replacing x by z and y by 0)—

$$\frac{\partial f}{\partial z} = i(2z - \frac{1}{z^2})$$

$$f(z) = i\left(z^2 + \frac{1}{z}\right) + c$$

Hence, $u = \text{Re}\left[i\left(z^2 + \frac{1}{z}\right) + k\right] = \text{Re}\left[i\left\{x^2 - y^2 + i(2xy) + \frac{x}{x^2+y^2} - i\frac{x}{x^2+y^2}\right\} + c\right]$

$$= -2xy + \frac{x}{x^2+y^2} + c$$

ans

Example— Find the analytic function $f(z) = u + iv$, whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2$.

Solution— Let $f(z) = u + iv$ is an analytic function. Hence, from the question—

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2$$

Now, $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ [by using cauchy – Riemann equation]

Hence, $\therefore \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = (3x^2 - 3y^2 + 6x) + i(-6xy - 6y)$

by using Milne-Thomson's method (replacing x by z and y by 0)—

$$\frac{\partial f}{\partial z} = (3z^2 + 6z) + i(0)$$

$$f(z) = z^3 + 3z^2 + ic$$

Hence, $v = \text{Im}[z^3 + 3z^2 + ic] = \text{Im}[x^3 + i3x^2y - 3xy^2 - iy^3 + 3x^2 - 3y^2 + 6xy + ic]$

$$\therefore v = 3x^2y - y^3 + c$$

Therefore $f(z) = (x^3 - 3xy^2 + 3x^2 - 3y^2) + i(3x^2y - y^3 + c)$ ans

Example— Find the analytic function $z = u + iv$, if $u - v = (x - y)(x^2 + 4xy + y^2)$.

Solution— Given that—

$$u - v = (x - y)(x^2 + 4xy + y^2)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2 \qquad \text{... (1)}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2 \qquad \text{... (2)}$$

On applying Cauchy-Riemann's theorem in equation (2)—

$$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 3x^2 - 6xy - 3y^2 \qquad \text{... (3)}$$

Now, from equations (1) and (3)—

$$\frac{\partial u}{\partial x} = 6xy \text{ and } \frac{\partial v}{\partial x} = -3x^2 + 3y^2$$

Again, $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 6xy + i(-3x^2 + 3y^2)$

By using Milne-Thomson's method—

$$\frac{\partial f}{\partial z} = i(-3z^2)$$

$$f(z) = -iz^3 + c$$

$$= -i(x^3 + i3x^2y - 3xy^2 - iy^3) + c$$

$$= (3x^2y - y^3 + c) + i(3xy^2 - x^3) \qquad \text{ans}$$

1.10.3 Exact Differential equation Method—

Let $f(z) = u + iv$ is an analytic function.

Case-I If $u(x, y)$ is given

We know that—

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{[By C-R equations]}$$

Let $dv = Mdx + Ndy$, therefore $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$

Again, $\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$

Therefore

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0 \qquad [\because u \text{ is a harmonic function}]$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

[Satisfies the condition for exact differential equation]

Hence,

$$v = \int_{y=const} \left(-\frac{\partial u}{\partial y}dx\right) + \int \left(\text{terms of } \frac{\partial u}{\partial x} \text{ not containg } x\right) dy + c$$

Case-II If $v(x, y)$ is given

We know that—

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy \qquad \text{[By C-R equations]}$$

Let $dv = Mdx + Ndy$, therefore $M = \frac{\partial v}{\partial y}$ and $N = -\frac{\partial v}{\partial x}$

Again, $\frac{\partial M}{\partial y} = \frac{\partial^2 v}{\partial y^2}$ and $\frac{\partial N}{\partial x} = -\frac{\partial^2 v}{\partial x^2}$

Therefore $$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\right) = 0 \qquad [\because v \text{ is a harmonic function}]$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ [Satisfies the condition for exact differential equation]

Hence,

$$u = \int_{y=const} \left(\frac{\partial v}{\partial y}dx\right) + \int \left(\text{terms of} \left(-\frac{\partial v}{\partial x}\right) \text{ not containg } x\right) dy + c$$

Example— If $u = x^3 - 3xy^2 + 3x + 1$, determine v for which $f(z)$ is an analytic function.

Solution— Given that— $u = x^3 - 3xy^2 + 3x + 1$

By using exact differential equation method, v can be written as—

$$v = \int_{y=const}\left(-\frac{\partial u}{\partial y}dx\right) + \int\left(\text{terms of } \frac{\partial u}{\partial x} \text{ not containg } x\right)dy + c$$

$$v = \int_{y=const}(6xydx) + \int(-3y^2 + 3)dy + c$$

$$v = 3x^2y - y^3 + 3y + c$$

Example— If $u = y^3 - 3x^2y$, determine v for which $f(z) = u + iv$ is an analytic function.

Solution— Given that— $u = y^3 - 3x^2y$

By using exact differential equation method, v can be written as—

$$v = \int_{y=const}\left(-\frac{\partial u}{\partial y}dx\right) + \int\left(\text{terms of } \frac{\partial u}{\partial x} \text{ not containg } x\right)dy + c$$

$$v = \int_{y=const}(-3y^2 + 3x^2)dx + \int(0)dy + c$$

$$v = -3xy^2 + x^3 + c$$

1.10.4 Shortcut Method

Case-I If $u(x, y)$ is given

If the real part of an analytic function $f(z)$ is given then—

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic$$

Where, c is a real constant.

Case-II If $v(x, y)$ is given

If the imaginary part of an analytic function $f(z)$ is given then—

$$f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) - iv(0,0) + c$$

Where, c is a real constant.

Example— If $f(z) = u + iv$ is an analytic function and $u(x, y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Solution— Given that $f(z)$ is an analytic function and it's real part is $u(x, y) = \frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Hence,

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ic$$

$$f(z) = 2\frac{\sin z}{\cosh\left(\frac{z}{i}\right) + \cos z} + ic$$

$$f(z) = 2\frac{\sin z}{\cosh(-iz) + \cos z} + ic \qquad [\cosh(-x) = \cosh x]$$

$$f(z) = 2\frac{\sin z}{\cosh(iz) + \cos z} + ic \qquad [\cosh(ix) = \cos x]$$

$$f(z) = 2\frac{\sin z}{\cos z + \cos z} + ic$$

$$f(z) = \tan z + ic$$

Example— Determine the analytic function $f(z)$, whose imaginary part is $3x^2y - y^3$.

Solution— Given that $v = 3x^2y - y^3$, since the complex variable function $f(z)$ is analytic, hence it is given by—

$$f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) - iv(0,0) + c$$

$$f(z) = 2i\left(3\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - \left(\frac{z}{2i}\right)^3\right) + c = 2i\left(\frac{3}{8i}z^3 + \frac{1}{8i}z^3\right) + c = 2i\left(\frac{1}{2i}z^3\right) + c$$

$$f(z) = z^3 + c$$

Example— Determine the analytic function $f(z)$, whose imaginary part is $\log(x^2 + y^2) + x - 2y$.

Solution— Given that $v = \log(x^2 + y^2) + x - 2y$

$$v_x = \frac{2x}{x^2+y^2} + 1, \qquad v_y = \frac{2y}{x^2+y^2} - 2$$

Let $f(z) = u + iv$, $\therefore \frac{\partial f}{\partial x} = u_x + iv_x = v_y + iv_x$ [By using C – R equations]

Hence, $f'(z) = -2 + i\left(\frac{2}{z} + 1\right)$

$$f(z) = -2z + 2i\log z + iz + c$$

$$f(z) = 2i\log z + (i - 2)z + c$$

Example— Find the orthogonal trajectories of the family of curves given by the equation—

$$\boldsymbol{e^x \cos y - xy = c.}$$

Solution— given curve is—

$$u = e^x \cos y - xy = c$$

The orthogonal trajectories to the family of curves $u = c_1$ will be $v = c_2$ such that $f(z) = u + iv$ becomes analytic. Hence, orthogonal trajectory $v = c_2$ is given by—

$$v = \int_{y=const} \left(-\frac{\partial u}{\partial y} dx\right) + \int \left(terms\ of\ \frac{\partial u}{\partial x}\ not\ containg\ x\right) dy + c$$

$$v = \int_{y=const} (e^x \sin y - x) dx + \int (-y) dy + c$$

$$v = e^x \sin y - \frac{x^2}{2} - \frac{y^2}{2} + c$$

Example— Show that the function $u(x,y)=e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x,y)$ and the analytic function $f(z)=u+iv$.

Solution— given that $u(x,y)=e^x\cos y$

Now, $u_x=e^x\cos y, \qquad u_{xx}=e^x\cos y$

$\qquad u_y=-e^x\sin y, \quad u_{yy}=-e^x\cos y$

Here, $u_{xx}+u_{yy}=0$, so **$u(x,y)$ is a harmonic function.**

Again, since $f(z)=u+iv$ is an analytic function, so $f(z)$ is given by—

$$f(z)=2u\left(\frac{z}{2},\frac{z}{2i}\right)-u(0,0)+ic$$

$$f(z)=2e^{\frac{z}{2}}\cos\left(\frac{z}{2i}\right)-1+ic \ =2e^{\frac{z}{2}}\cos\left(\frac{-iz}{2}\right)-1+ic \ =2e^{\frac{z}{2}}\cos\left(\frac{iz}{2}\right)-1+ic$$

$$f(z)=\left(2e^{\frac{z}{2}}\right)\left(\frac{1}{2}\right)\left(e^{\frac{z}{2}}+e^{\frac{-z}{2}}\right)-1+ic$$

$$f(z)=(e^z+1)-1+ic$$

$$\boldsymbol{f(z)=e^z+ic}$$

Now, $f(z)=e^{x+iy}+ic \ =e^x e^{iy}+ic$

$$f(z)=e^x(\cos y+i\sin y)+ic$$

$$f(z)=e^x\cos y+i\,(e^x\sin y+c)$$

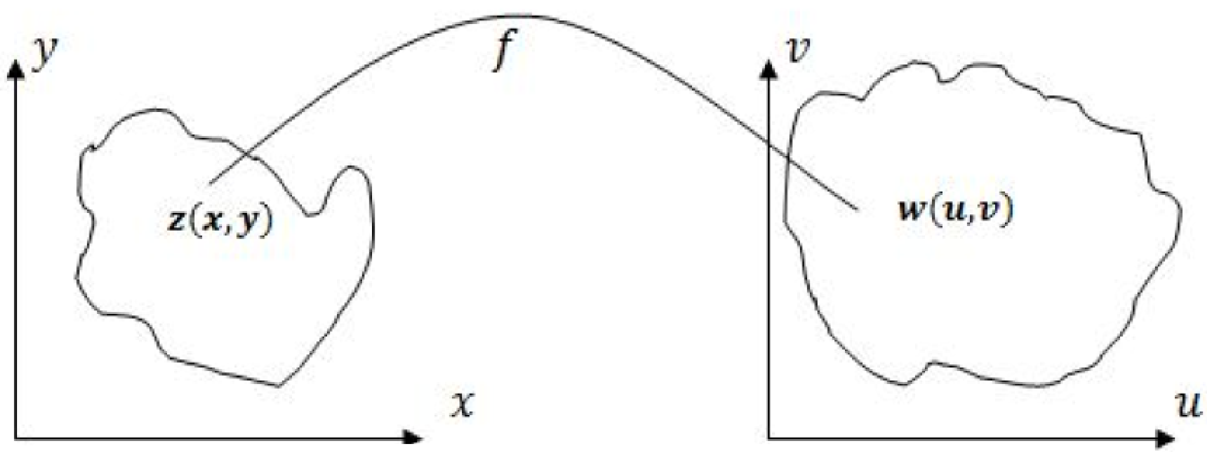
Hence, $v(x,y)=e^x\sin y+c$

1.11 Conformal Mapping—

1.11.1 Mapping—

Mapping is a mathematical technique used to **convert** (or map) one mathematical problem and its solution into another. It involves the study of **complex variables**.

Let a complex variable function $z = x + iy$ define in Z- plane have to **convert** (or map) in another complex variable function $f(z) = w = u + iv$ define in w- plane. This process is called as **Mapping.**



Mapping

1.11.2 Conformal Mapping—

Conformal Mapping is a mathematical technique used to **convert** (or map) one mathematical problem and its solution into another preserving both angles and shape of infinitesimal small figures but not necessarily their size.

The process of **Mapping** in which a complex variable function $z = x + iy$ define in Z- plane mapped to another complex variable function $f(z) = w = u + iv$ define in w- plane preserving the angles between the curves both in magnitude and sense is called as **conformal mapping.**

The **necessary condition** for **conformal mapping**— if $w = f(z)$ represents a conformal mapping of a domain D in the z −plane into a domain D of the w −plane then $f(z)$ is an analytic function in domain D.

Application of Conformal mapping— A large number of problems arise in fluid mechanics, electrostatics, heat conduction and many other physical situations.

There are different types of conformal mapping. Some standard conformal mapping are mentioned below—

1. Translation
2. Rotation
3. Magnification
4. Inversion
5. Reflection

1.11.2.1 Translation— A conformal mapping in which every point in z –plane is translated in the direction of a given vector is known as Translation.

Let a complex function $z = x + iy$ translated in the direction of the vector $\alpha = a + ib$, then that translation is given by— $w = z + \alpha$.

Example— Determine and sketch the image of $|Z| = 1$ under the transformation $w = z + i$.

Solution— Let $w = u + iv$

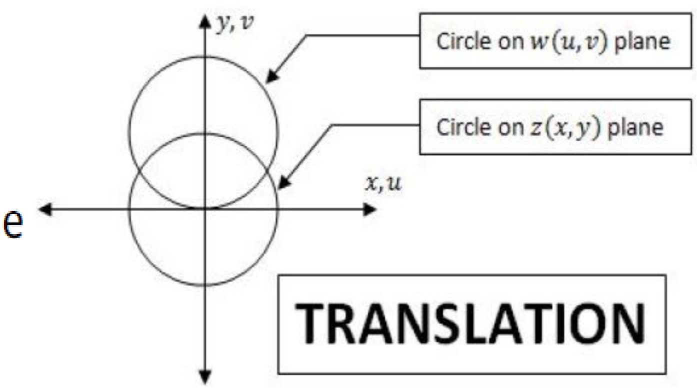
$\therefore u + iv = x + iy + i = x + i(y + 1)$

Hence, $x = u$ and $y = v - 1$

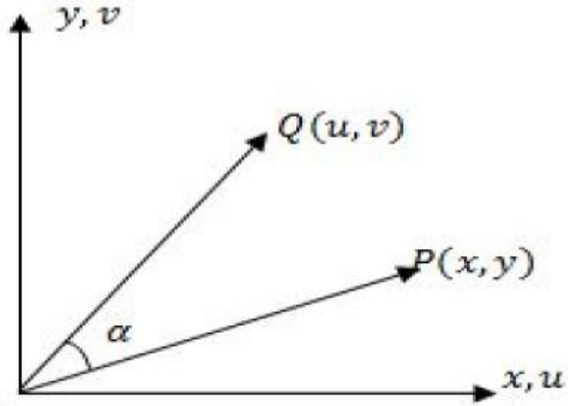
Again, from the question $|Z| = 1$

$\therefore x^2 + y^2 = 1$

$u^2 + (v - 1)^2 = 1$, which represent a circle in (u, v) plane



1.11.2.2 Rotation— A conformal mapping in which every point on z −plane, let $\mathrm{P}(x, y)$ such that OP is rotated by an angle α in anti-clockwise direction mapped into w −plane given by $w = e^{i\alpha} z$ is called as Rotation.



ROTATION

Example– Determine and sketch the image of rectangle mapped by the rotation of $45°$ in anticlockwise direction formed by $x = 0, y = 0, x = 1$ and $y = 2$.

Solution– given rectangle is–

$$x = 0, y = 0, x = 1 \text{ and } y = 2$$

Let the required transformation is–

$$w = e^{i\frac{\pi}{4}} z$$

$$w = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(x + iy)$$

$$w = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x + iy)$$

$$w = \frac{x-y}{\sqrt{2}} + i\frac{x+y}{\sqrt{2}}$$

$$u + iv = \frac{x-y}{\sqrt{2}} + i\frac{x+y}{\sqrt{2}}$$

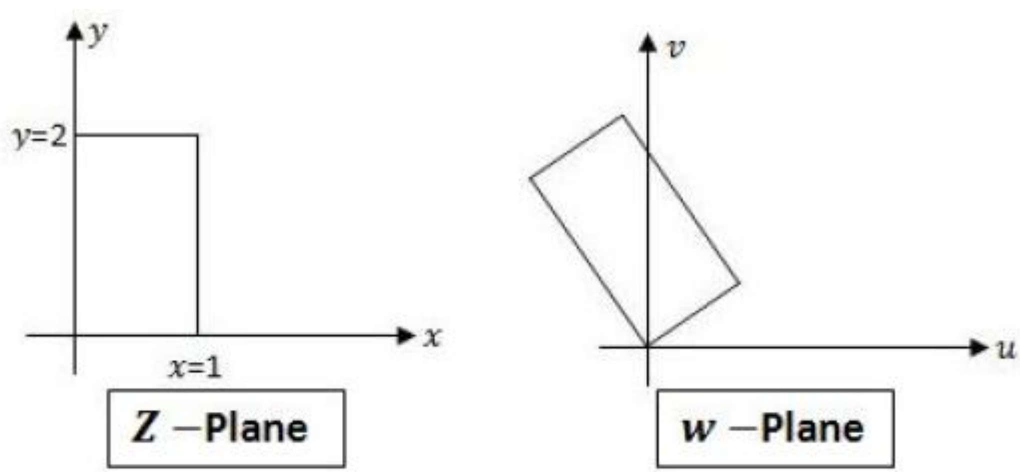
$$\therefore u = \frac{x-y}{\sqrt{2}} \text{ and } v = \frac{x+y}{\sqrt{2}}$$

Now, for $x = 0$— $\quad u = \dfrac{-y}{\sqrt{2}}, \quad v = \dfrac{y}{\sqrt{2}} \Rightarrow u + v = 0$

For $y = 0$— $\quad u = \dfrac{x}{\sqrt{2}}, \quad v = \dfrac{x}{\sqrt{2}} \Rightarrow u - v = 0$

For $x = 1$— $\quad u = \dfrac{1 - y}{\sqrt{2}}, \quad v = \dfrac{1 + y}{\sqrt{2}} \Rightarrow u + v = \sqrt{2}$

or $y = 2$— $\quad u = \dfrac{x - 2}{\sqrt{2}}, v = \dfrac{x + 2}{\sqrt{2}} \Rightarrow u - v = -2\sqrt{2}$



1.11.2.3 Magnification (Scaling) — A conformal mapping in which every point on z −plane mapped into w −plane given by $w = cz$ ($c > 0$ is real) is called as magnification. In this process mapped shape is either stretched($c > 1$) or contracted ($0 < c < 1$) in the direction of Z.

Example— Determine the transformation $w = 2z$, where $|z| = 2$.

Solution— given that $|z| = 2 \Rightarrow x^2 + y^2 = 4$, represents a circle on z −plane having center at origin and radius equal to 2-units.

Let the required transformation is—

$$w = 2z = 2(x + iy) = 2x + i2y$$
$$u + iv = 2x + i2y$$

Hence, $x = \frac{u}{2}$ and $y = \frac{v}{2} \Rightarrow \left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 = 4 \Rightarrow u^2 + v^2 = 16$ represent a circle on w −plane having center at origin and radius equal to 4-units.

1.11.2.4 Inversion— A conformal mapping in which every point on z −plane mapped into w −plane given by $w = \frac{1}{z}$ is called as inversion.

Example— Determine the transformation $w = \frac{1}{z}$, where $|z| = 1$.

Solution— given that— $|z| = 1 \Rightarrow x^2 + y^2 = 1$ ... (1)

Again, given that

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$u + iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = x - iy \qquad [x^2 + y^2 = 1]$$

Hence, $u = x$ and $v = -y$

Now, from equation (1)—

$$u^2 + v^2 = 1$$

1.11.2.5 Reflection — A conformal mapping in which every point on z −plane mapped into w −plane given by $w = \bar{z}$. This represents the reflection about real axis.

Example— Determine and sketch the transformation $w = \bar{z}$, where $|z - (2 + i)| = 4$.

Solution— given that—

$$|z - 2| = 4$$

$$|(x - 2) + i(y - 1)| = 4$$

$$(x - 2)^2 + (y - 1)^2 = 16 \qquad \text{... (1)}$$

Again, from the question—

$$w = \bar{z} = x - iy$$

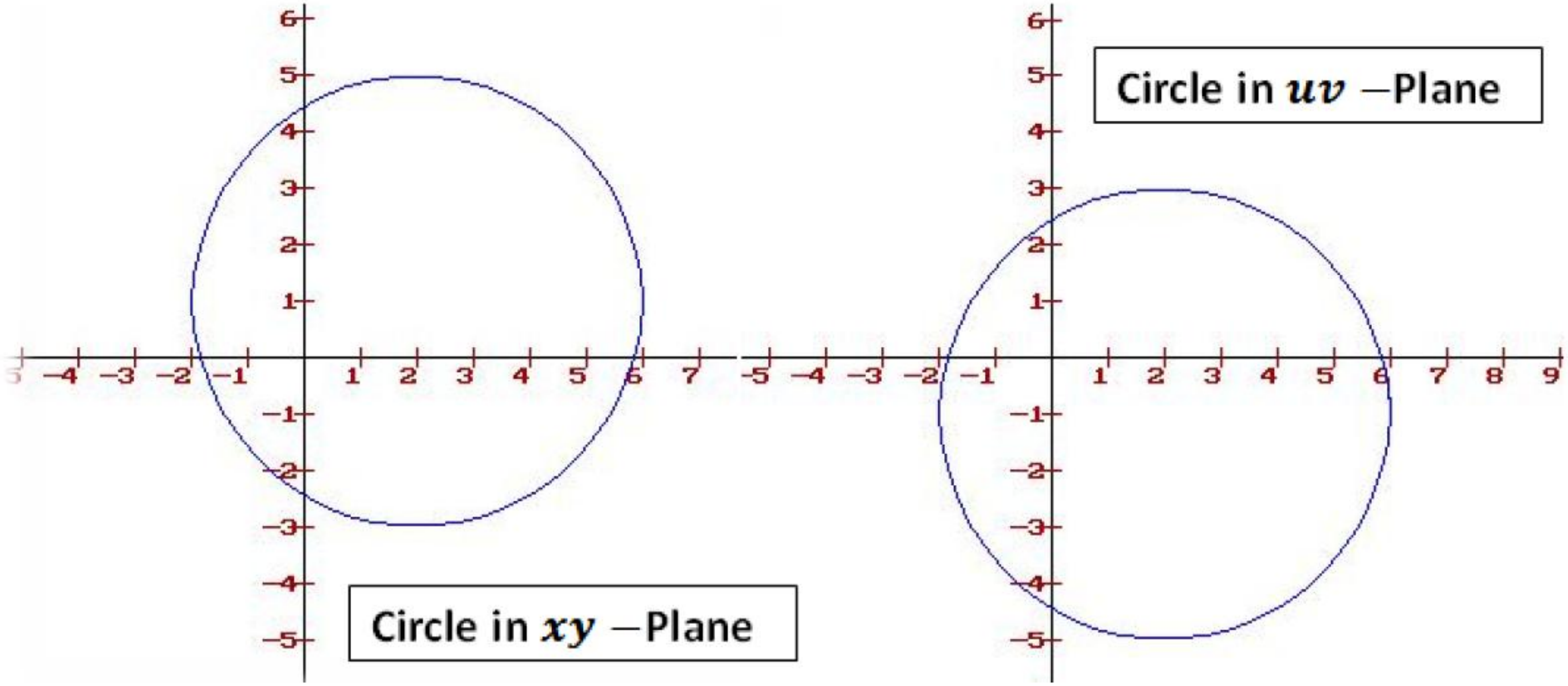
$$u + iv = x - iy$$

Hence, $x = u$ and $y = -v$

Now, from equation (1)—

$$(u - 2)^2 + (v + 1)^2 = 16$$

Sketch—



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Thanks