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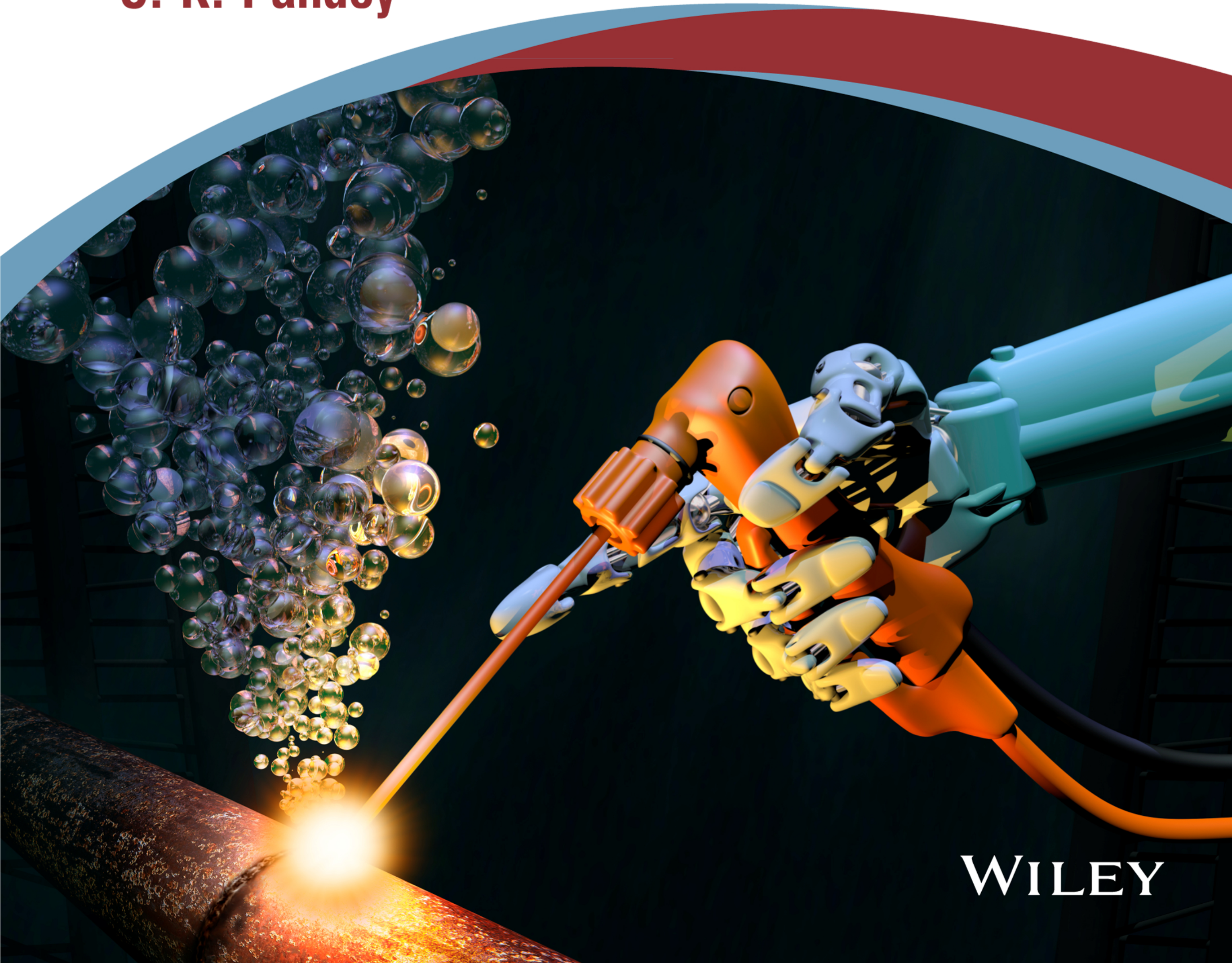
Engineering Physics

Theory and Practical

A. K. Katiyar

C. K. Pandey



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Theory and Practical

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Preface

It is our privilege to bring out the present book titled "*Engineering Physics – Theory and Practical*" for engineering students of U. P. Technical University, Lucknow. It is well known that Physics is not only the basic subject from which all the branches of engineering are derived but is also used in most of the technological developments and their advancements. In this regards, it becomes the compulsory course for all engineering graduated students. Since U. P. Technical University has revised the syllabus of Engineering Physics from academic session 2013–14, therefore the present textbook is an attempt to fulfil the needs of all engineering students according to the new revised syllabus.

The subject matter in the book has been presented in easy, effective and systematic way starting from basic concepts for the sake of continuity and better understanding of the subject. The whole subject matter has been divided into theory and practical sections as per the UPTU syllabus.

The University runs two courses of Engineering Physics (NAS-101 and NAS-201) in first and second semester, respectively. Therefore, the theory section of the book contains entire syllabus of both courses presented in 14 chapters according to the papers. Chapters 1–7 contain the syllabus of first paper (NAS-101) and Chapters 8–14 contain the syllabus of second paper (NAS-201).

Organization of the Book

NAS-101

1. Chapter 1 of the book explains the relativistic mechanics (Unit-I).
2. Chapter 2 describes the fundamentals of modern physics (Unit-II).
3. Physical optics of Unit-III Interference, Diffraction and Polarization is presented through Chapters 3, 4 and 5, respectively.
4. Chapters 6 and 7 cover laser, fibre-optics and holography of Unit-IV.

NAS-201

1. Chapter 8 explains the crystal structure and X-ray diffraction of Unit-I.
2. Chapters 9 and 10 contain dielectric and magnetic properties of materials of Unit-II.
3. Unit-III consists the electromagnetic theory describes through Chapter 11.
4. Some technologically important materials such as semiconductors, superconductors and nanomaterials of Unit-IV are expressed through Chapters 12, 13 and 14, respectively.

Lab Manual

The practical section of book contains detailed theory, method, observation table and question and answer for viva-voce. It provides complete information on all experiments prescribed as per UPTU syllabus.

Salient Features

1. Every topic is written and explained in a systematic and step-wise manner so that anyone can understand the subject without any difficulties. The language used is so lucid and comprehensive that a student lacking good knowledge of the subject can also be equally benefitted.
2. Review questions (related to the topics) and important points are provided at the end of the each chapter for further exposure and memories the contents.
3. Numerical problems with step-by-step solutions are provided in each chapter for understanding and practice.
4. **Lab Manual:** The experiments and their viva-voce aspect are incorporated in a very simple and systematic way in this book.

Acknowledgements

We owe a deep sense of gratitude to Manisha Bajpai, PDF Allahabad University, Allahabad and Dr. S. A. Warsi SRMCEM, Lucknow for providing us enough opportunities for interacting with them on the topics covered in the present book. We are thankful to publishers WILEY for providing constant support during the work and bringing the book in such a nice form. Although we have made our best efforts for error-free book, we would be glad to know any misprint omission that has crept in the printed matter or any genuine and constructive suggestions for improving the quality of the present textbook.

Authors
April 03, 2015

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Contents

Chapter 1 Relativistic Mechanics 1

Chapter 2 Wave Mechanics 33

Chapter 3 Wave Optics: Interference 65

Chapter 4 Diffraction of Light 89

Chapter 5 Polarization of Light 113

Chapter 7 Fiber Optics and Holography 139

Chapter 8 Crystal Structure 155

Chapter 9 Dielectrics 181

Chapter 10 Magnetic Properties of Materials 199

Chapter 11 Electromagnetics 211

Chapter 12 Semiconductors 241

Chapter 13 Superconductivity 257

1

Relativistic Mechanics

LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Frame of reference.
- Michelson–Morley experiment.
- Einstein postulates.
- Lorentz transformation equations.
- Length contraction and time dilation.
- Addition of velocities.
- Variation of mass with velocity.
- Mass energy equivalence.

1.1 Introduction

Theory of relativity is nothing but the branch of physics which deals with the motion of material bodies relative to each other and rest. It is also known as relativistic mechanics and is classified into two parts: *Special theory of relativity* and *general theory of relativity*. Special theory of relativity deals with the relative motion which remains at constant speed or at rest. General theory of relativity deals with the arbitrary relative motion which is not at constant speed and may be accelerated with respect to the material bodies. Before theory of relativity, the motion was described by Newton's law under classical mechanics. In the very beginning it was thought that classical mechanics is applicable to all types of speed, but later the experimental fact revealed that classical mechanics is not applicable to bodies moving with the velocity comparable to the velocity of light. In classical mechanics, space and time are separable, or in other words, space and time are absolute and the transformations connecting the space–time coordinates of a particle are the Galilean transformations. These transformations are valid as far as Newton's laws are concerned, but fail for bodies moving with the velocity of light.

The principle of relativity, when applied to the electromagnetic phenomena, asserts that the speed of light in vacuum is a constant of nature. This statement has been experimentally confirmed by various investigators and led Einstein to formulate the special theory of relativity in 1905. According to this theory, *everything in the universe is relative, nothing is absolute, all rest and motions are relative, position and time are relative, etc*. In other words, one can say that space and time are not independent of each other and the correct transformation equations are Lorentz transformations. We can understand it through following examples:

1. Consider an observer sitting in a moving train looking at distant stationary objects like trees or buildings. All these appear moving in the reverse direction of the motion of the train. It is easy to realize for a person standing outside the train that the observer of the train is moving in a particular direction, while for observer, the standing person would appear to move in a direction opposite to his own direction. It all concludes that the motion is relative.

2. If we compare the time in India during chat with a friend in China, we find that the time in India is different from the time in China. Hence, time is also relative.
3. Consider two people facing each other, standing on the opposite banks of the river and watching a boat moving in the river. For the first person the boat is towards his right while for the second one it is to his left. This clearly indicates that position is relative.

These examples make it clear that what an observer observes depends on his state or his *frame of reference*.

In this chapter we shall limit ourselves to the special theory of relativity, so first we will discuss frame of reference, ether hypothesis, Michelson–Morley experiments, the Galilean transformations and their failures and then we shall deduce the Lorentz transformations. After that, we will discuss the consequences of Lorentz transformations like length contraction, time dilation, velocity addition, etc.

1.2 Some Important Terms

Some common terms which are frequently used in relativistic mechanics are as follows:

1. **Particle:** A particle is a small piece of matter, having practically no linear dimension, but only a position at a point. It is characterized by its mass and charge. Example: electron, proton, photon, etc.
2. **Observer:** A person who locates, records, measures and interprets an event is called an observer. The observer draws his interferences about the events on the basis of his observations.
3. **Event:** In relativity, an event implies anything that occurs suddenly or instantaneously at a point in space. It involves both a position and a time of occurrence.

1.3 Frame of Reference

In order to specify the location of a point object in space, we require a coordinate system. A *system of coordinate axes which defines the position of a particle or specifies the location of an event is called a frame of reference*. The simplest frame of reference is the *Cartesian system of coordinates* in which the location of a point is specified by the three (x, y and z) coordinates. For complete information about an event we must not only know about its locations but also its correct time of occurrence. Therefore, in addition to the three space coordinates, we need one more coordinate – time t – of its occurrence. A frame of reference having four coordinates, x, y, z and t is referred to as a *space–time frame*. If we are to describe events, our first step should be to establish a frame of reference. The frames of reference are classified into two groups:

1. Inertial frames of reference.
2. Non-inertial frames of reference.

1.3.1 Inertial Frames of Reference

In inertial frames of reference, bodies obey Newton's law of inertia and other laws of Newtonian or classical mechanics. In such a frame of reference, if no net external force acts upon a body, the body will move with zero acceleration – that is moving with a constant velocity. So it is also known as unaccelerated frames of reference. In other words, *all those frames of reference which are either stationary relative to each other or are in uniform motion are called the inertial frames*. Newton assumed that a frame of reference fixed with respect to the stars is an inertial system. A rocket ship drifting in outer space without spinning and with its engine cut off provides an ideal inertial system. Any set of axes moving at uniform velocity with respect to the earth, as in a train, ship or airplane, will be inertial because motion at uniform velocity does not introduce acceleration. *The special theory of relativity, which we consider here, deals only with the description of events by observers in inertial frames of reference*.

1.3.2 Non-Inertial Frames of Reference

The frames of reference in which Newton's law is not valid is said to be non-inertial. In other words, the accelerated frames of reference are called non-inertial. A system of axes which accelerates with respect to the earth, such as one fixed to a spinning merry-go-round or to an accelerating car, is non-inertial system. A particle acted on by zero net external force will not move in straight line with constant speed according to an observer in such non-inertial system.

1.4 Earth: Inertial or Non-Inertial Frame of Reference?

We know that earth is not only rotating about its own axis but also orbiting around the sun. So accelerations are present in both motions due to centripetal force. In this sense, one can say that earth is non-inertial frame of reference. However, in this study the speed of light is of the order of 2.99×10^8 m/sec and the earth is moving at a speed of 30 km/sec. So the effects of rotation and revolution of earth may be ignored. Thus, earth or any frame of reference set-up on earth is regarded as an inertial frame of reference.

1.5 Ether Hypothesis

According to Maxwell, light waves are basically electromagnetic waves which are propagated through free space or vacuum with the speed of 2.99×10^8 m/sec. Till 19th century, all the waves known to mankind were mechanical waves, which required a material medium for their propagation. *It was assumed that the entire space of the universe including vacuum is filled by a hypothetical medium called ether which is rigid, invisible, massless, perfectly transparent*. On the necessity of the medium, scientists tried to detect and understand the relative motion of physical bodies with respect to ether. Many experiments were conducted in this direction; the most famous among them being Michelson–Morley experiment, which we will discuss next. However, negative results of this experiment ruled out the existence of this hypothetical ether medium.

1.6 Michelson–Morley Experiment

In 1887, Albert Michelson and Edward Morley carried out an experiment to detect the motion of the earth relative to ether medium at rest using Michelson interferometer. Michelson was awarded the Noble prize in physics for this experiment. *The main objective of this experiment was to confirm the existence of ether*. If we imagine ether to be fixed with respect to the sun, then the earth moves through the ether at a speed of 30 km/sec in different directions in different seasons. Therefore, the time taken by the light to travel equal distance in different directions would be different. So we have to find this time difference from which the relative velocity between the ether and the earth could be estimated.

The arrangement for Michelson–Morley experiment is shown in Fig. 1. A beam of light from the source S is incident upon a 45° inclined glass plate P. It splits into two components: One is reflected and other is refracted. These beams travel at right angles to each other and are normally incident on mirrors M_1 and M_2 placed at equal distances $PM_1 = PM_2 = L'$ from the glass plate P. After reflections from the mirrors, the two beams interfere at point P. The interference fringes are observed in the telescope. If the apparatus were at rest, the two beams would take the same time to return to P.

Let us consider that earth along with the apparatus moves with a velocity v in ether. Suppose c is the velocity of light through the ether. When light goes from P to M_1, the relative velocity of light is $c - v$. From M_1 to P the relative velocity is $c + v$. Finally, from either P to M_2 or M_2 to P, the relative velocity of light is $(c^2 - v^2)^{1/2}$. Thus the time required by light to go along the parallel path from P to M_1 and back to P, as measured by the observer O', is

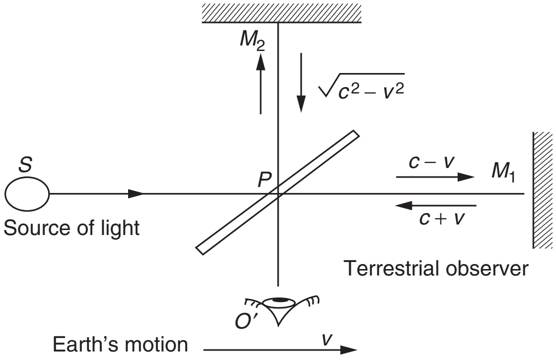


Figure 1 Michelson–Morley experiment.
Note: $PM_1 = PM_2 = L'$.

$$t_1 = \frac{L'}{c-v} + \frac{L'}{c+v} = \frac{2L'c}{c^2 - v^2} = \frac{2L'/c}{1-(v^2/c^2)}$$

However, the time required to go along the perpendicular path from P to M_2 and back to P, as measured by O', is

$$t_2 = \frac{2L'}{(c^2 - v^2)^{1/2}} = \frac{2L'/c}{[1-(v^2/c^2)]^{1/2}}$$

Hence, the time difference between the times of the travel of the two beams is

$$\Delta t = t_1 - t_2 = \frac{2L'/c}{1-(v^2/c^2)} - \frac{2L'/c}{[1-(v^2/c^2)]^{1/2}} = \frac{2L'}{c}[\{1-(v^2/c^2)\}^{-1} - \{1-(v^2/c^2)\}^{-1/2}]$$

Using binomial theorem and neglecting higher terms, we get

$$\Delta t = \frac{2L'}{c}\left[\left(1 + \frac{v^2}{c^2} + \cdots\right) - \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots\right)\right] = \frac{2L'}{c}\left[\frac{1}{2}\frac{v^2}{c^2}\right] = \frac{L'v^2}{c^3}$$

Now the corresponding path difference is

$$\Delta = c\Delta t = c\left(\frac{L'v^2}{c^3}\right) = \frac{L'v^2}{c^2}$$

Finally, the whole apparatus is turned through 90° so that the path PM_1 becomes longer than the path PM_2 by an amount $L'v^2/c^2$. As a result, a path difference of same amount in opposite direction is introduced so that the total path difference between the two rays becomes $2L'v^2/c^2$. Thus, the fringe shift is

$$\Delta n = \frac{\text{Path difference}}{\lambda} = \frac{2L'v^2}{c^2} \cdot \frac{1}{\lambda}$$

Now put the value $L' = 11$ m (taken by Michelson and Morley), $\lambda = 6000$ Å and $v = 3 \times 10^4$ m/sec. We get

$$\Delta n = \frac{2L'v^2}{c^2} \cdot \frac{1}{\lambda} = 0.4$$

A fringe shift of this amount is readily detected with the apparatus. Michelson and Morley were surprised to see that no shift in the fringe was observed even when the interferometer was rotated through 90°. *This indicates that the relative velocity between the earth and the ether is zero*. The experiments since then have been repeated several times under different circumstances but always the same negative result was obtained.

1.6.1 Explanation of the Negative Results of Michelson–Morley Experiment

There are many explanations and interpretations of negative results of the Michelson–Morley experiment to defend the concept of stationary hypothetical medium ether. But these have failed. Some of them are presented here:

1. **Ether–Drag hypothesis:** The moving earth drags the ether with it. Hence, there is no relative motion between the two and hence no shift is observed.
2. **Fitzgerald–Lorentz contraction hypothesis:** This hypothesis suggests that all material bodies are contracted in the direction of motion relative to stationary ether by a factor $[1-(v^2/c^2)]^{1/2}$. As a result, the time taken by the two rays in travelling towards the mirrors M_1 and M_2 would be equal. So $\Delta t = 0$, which gives that there is no path difference and, hence, no shift would be expected.
3. **Principle of constancy of speed of light:** The velocity of light is constant and does not depend upon the motion of the source, observer or the medium.

1.6.2 Conclusions of Michelson–Morley Experiment

The followings are the main conclusions of the Michelson–Morley experiment:

1. The velocity of light is same in all directions.
2. The effects of the presence of ether are undetectable.

1.7 Einstein's Postulates of Special Theory of Relativity

Einstein drew two important conclusions for the formulation of the special theory of relativity. These are known as the postulates of the special theory of relativity. These postulates are:

1. *All the laws of physics are the same in every inertial frame of reference*. This postulate implies that there is no experiment, whether based on the laws of mechanics or the laws of electromagnetism, from which it is possible to determine whether or not the frame of reference is in a state of uniform motion.
2. *The speed of light is independent of the motion of its source*. Einstein was inspired to make these postulates through his study of the properties of Maxwell's equations and not by the negative results of the Michelson–Morley experiment, of which he was apparently only vaguely aware. It is this postulate that forces us to reconsider what we understand by space and time.

One immediate consequence of these two postulates is that the speed of light is the same in all inertial frames of reference.

1.8 Galilean Transformation

At any instant, the coordinates of a point or particle in space will be different in different coordinate systems. *The equations which provide the relationship between the coordinates of two reference systems are called transformation equations*. In Newtonian mechanics where the speed of the observer or object is very small compared to the speed of light, these relevant transformation equations are called *Galilean transformation equations or Galilean transformations*. Galilean transformations are used to transform the coordinates of position and time from one inertial frame to another.

In order to obtain the Galilean transformation equations, consider two frames of reference S and S' with axes (x, y, z) and (x', y', z'), respectively. The frame S' is moving with a uniform velocity v along the x-axis. At $t = 0$, the two frames coincided which means that the axis of S and S' overlapped. At any time t, the x-coordinate of point P in S exceeds that in S' by vt, the distance covered by S' in time t in the positive x direction as shown in Fig. 2.

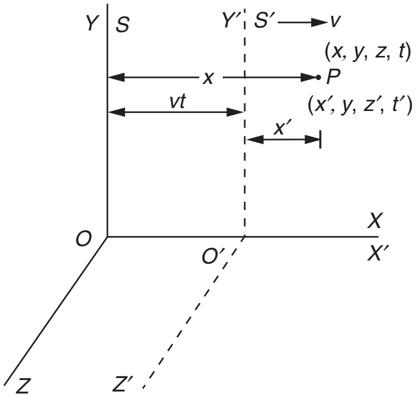


Figure 2 Motion of frame S' with constant velocity v.

Therefore, the observed coordinates in the two frames are given by the following transformation equations:

$$x' = x - vt; \quad y' = y; \quad z' = z; \quad t' = t \tag{1.1}$$

The set of equations (1.1) are known as Galilean transformations. We can consider that frame S is moving with velocity $-v$ along the negative x-axis with respect to frame S'. Then the transformation equations from S' to S are as follows:

$$x = x' + vt; \quad y = y'; \quad z = z'; \quad t = t' \tag{1.2}$$

These are known as *inverse Galilean transformation equations*.

The general form of transformation equations is $r' = r - vt$ and that of inverse transformation equations is $r = r' + vt$.

The other assumption, regarding the nature of the space, is that the distance between two points is independent of any particular frame of reference. For this purpose, we consider a rod of length L in the frame S with the end coordinates x_1 and x_2. Then using Galilean transformation equations we have

$$L = x_2 - x_1$$

If at the same time the end coordinates of the rod in S' are x_1' and x_2', then $L' = x_2' - x_1'$. According to Eq. (1.1) for any time t, we have

$$x_2' - x_1' = x_2 - x_1$$

Therefore, $L' = L$. *Thus, the length between two points is invariant under Galilean transformations.*

Differentiating the transformation Eq. (1.1), we get velocity transformation equations from S to S'. These equations are as follows:

$$\left.\begin{aligned} u_x' &= u_x - v \\ u_y' &= u_y \\ u_z' &= u_z \end{aligned}\right\} \frac{dx'}{dt} = u_x'; \ \frac{dx}{dt} = u_x \tag{1.3}$$

Thus, velocity is not invariant under Galilean transformations. In general, the velocity transformation is given by

$$\frac{dr'}{dt} = \frac{dr}{dt} - v \quad \text{or} \quad u' = u - v$$

while the inverse velocity transformation is given by

$$u = u' + v$$

The acceleration transformation equations are obtained by differentiating Eq. (1.3). We have

$$a_x' = a_x; \quad a_y' = a_y; \quad a_z' = a_z \tag{1.4}$$

Thus, the acceleration is invariant under Galilean transformations.

In classical physics the mass is also unaffected by the motion of the reference frame. Thus, the product ma will be the same for all inertial observers. According to Newton's second law

$$F = ma \quad \text{then} \quad F' = ma' \quad \text{[using Eq. (1.4)]}$$

or

$$F = F'$$

Similarly, we can prove that

$$F'_x = F_x, \ F'_y = F_y, \ F'_z = F_z,$$

Thus Newton's law is valid in both S and S', that is S and S' are inertial frame of references. *We may also say that Newton's laws of motion are invariant under Galilean transformations.*

1.8.1 Failure of Galilean Transformation

The principle of relativity asserts that the laws of physics are invariant in all inertial systems, moving with constant relative velocity. It is to be mentioned that the Galilean transformations satisfy principle of relativity as for as Newton's laws of motion are concerned, but as we shall see later that these transformation do not satisfy this principle for propagation of electromagnetic waves. In spite of this, if we measure the speed of light along the x-direction in the frame S which is to be c, then in the moving frame S' it will be $c' = c - v$. This violates the second postulate of the special theory of relativity. Therefore, a different transformation is required to satisfy the postulates of special theory of relativity.

1.9 Lorentz Transformations

A transformation that changes space–time coordinates (x, y, z, t) *into* (x', y', z', t') *in such a way that the speed of light is constant in all unaccelerated coordinates system is satisfied was first obtained by Lorentz and is hence called Lorentz transformation.*

In deriving this transformation, we will eventually make use of the constancy of the speed of light, but first we will derive the general form that the transformation law must take purely from kinematic/symmetry considerations. Refer Fig. 2 where we considered two inertial frames S and S', S' is moving with a velocity v relative to S. Let us consider that the origins of both frames coincide when the times on the clocks in each frame of reference are set to read zero, that is $t = t' = 0$. Now consider an event that occurs at the point $P(x, y, z, t)$ as measured in S. The same event occurs at (x', y', z', t') in S'. In our new transformation, the measurement in the x-direction made in frame S must be linearly proportional to that made in S'. That is

$$x' = \gamma(x - vt) \tag{1.5}$$

where γ is the proportionality constant, which does not depend upon either x or t but may be a function of v. As the laws of physics are same in both frames S and S' (first postulate), therefore the corresponding equation of x in terms of x' and t' will have similar nature except that $-v$ replaces v, so that

$$x = \gamma(x' + vt') \tag{1.6}$$

where $t \neq t'$. Now substituting the value of x' from Eq. (1.5) into Eq. (1.6), we have

$$x = \gamma[\gamma(x - vt) + vt']$$

$$\Rightarrow x = \gamma^2 (x - vt) + \gamma vt'$$

$$\Rightarrow (1 - \gamma^2)x + \gamma^2 vt = \gamma vt'$$

$$\Rightarrow t' = \gamma t + \left(\frac{1-\gamma^2}{\gamma v}\right)x = \gamma t - \frac{\gamma x}{v}\left(1 - \frac{1}{\gamma^2}\right) \tag{1.7}$$

The value of γ can be evaluated with the help of second postulate. Let a light signal be given at the origin O at time $t = 0$, $t' = 0$; this means O and O' coincide. The signal travels with a speed c which is same in both the frames:

$$x = ct \text{ and } x' = ct' \text{ (position in } S \text{ and } S' \text{ frame, respectively)}$$

Substituting these values of x and x' in Eqs. (1.5) and (1.6), we get

$$ct' = \gamma t(c - v) \text{ and } ct = \gamma t'(c + v)$$

After multiplying both these equations with each other, we get

$$c^2 = \gamma^2(c^2 - v^2) \quad \Rightarrow \gamma^2 = \frac{c^2}{(c^2 - v^2)} \quad \Rightarrow \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}} \tag{1.8}$$

Now, substituting the value of γ from Eq. (1.8) in Eq. (1.5), we get

$$x' = \frac{x - vt}{\sqrt{1 - (v^2/c^2)}} \tag{1.9}$$

Squaring and rearranging Eq. (1.9), we get

$$\gamma^2 = \frac{1}{1-(v^2/c^2)} \quad \text{or} \quad \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Substituting the value of $1-(1/\gamma^2)$ in Eq. (1.7) we get

$$-t' = \gamma t - \frac{\gamma x}{v}\frac{v^2}{c^2} \quad \text{or} \quad t' = \frac{t-(xv/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.10}$$

Due to the relative motion of the two reference frames is in the *x*-direction, it is reasonable to expect that all distances measured at right angles to the *x*-direction will be same in both the frame S and S', that is

$$y = y' \quad \text{and} \quad z = z' \tag{1.11}$$

Hence the **Lorentz transformation equations** for space and time are

$$x' = \frac{x-vt}{\sqrt{1-(v^2/c^2)}}; \quad y = y'; \quad z = z' \text{ and } t' = \frac{t-(xv/c^2)}{\sqrt{1-(v^2/c^2)}}$$

1.10 Inverse Lorentz Transformations Equations

In order to obtain the space and time coordinates (x, y, z, t) in frame S, we replace v by $-v$ in Eqs. (1.9), (1.10) and (1.11). The transformation equations are as follows:

$$x = \frac{x' + v\,t'}{\sqrt{1-(v^2/c^2)}} \tag{1.12}$$

$$y = y' \quad \text{and} \quad z = z' \tag{1.13}$$

$$t = \frac{t' + (xv/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.14}$$

These equations are known as **inverse Lorentz transformation equations**.

Since, nothing *can move with a velocity greater than the velocity of light* this means that v should always be less than c. Therefore, the Lorentz transformation equations are reduced to the Galilean transformation equations.

When $v << c$, that is $\sqrt{1-(v^2/c^2)} \approx 1$ and $t-(xv/c^2) \approx t$, then

$$x' = x - vt; \; y = y'; \; z = z'; \; t = t'$$

From a practical point of view, at low speeds, there is no difference between the Lorentzian and Galilean transformations and we use the later in most of the problems. However, for very fast particles having velocities comparable to the velocity of light, for example electrons in the atoms, cosmic ray particles, we must use the Lorentz transformation.

1.11 Consequences of Lorentz Transformations

Let us now discuss the consequences of Lorentz transformations regarding the length of the bodies and the time intervals between given events.

1.11.1 Length Contraction

The first of the interesting consequences of the Lorentz transformation is that the length no longer has an absolute meaning: The length of an object depends on its motion relative to the frame of reference in which its length is being measured. Lorentz and Fitzgerald observed that *when an object moves with a velocity* v (*comparable with velocity of light*) *relative to a stationary observer, its length appears to be contracted by a factor* $\sqrt{1-(v^2/c^2)}$, *in the direction of its motion whereas its other dimensions perpendicular to the direction of the motions are unaffected*. This decrease in length in the direction of the motion is called the *length contraction*.

In order to obtain the expression for length contraction let us consider a rod parallel to the x-axis moving with a velocity v relative to a frame of reference S. This rod is stationary relative to a frame of reference S' which is also moving with a same velocity v relative to S.

As the rod is stationary in frame S', the ends of the rod will have coordinates x_1' and x_2' which remain fixed as functions of the time in S'. The length of the rod, as measured in S', is then

$$l_0 = x_2' - x_1' \tag{1.15}$$

where l_0 is known as the proper length of the rod, that is l_0 is its length as measured in a frame of reference in which the rod is stationary. Now we want to measure the length of the rod in frame S. In order to do this, we measure the x-coordinates of the two ends of the rod at the same time t by the clock in S. Suppose x_2 and x_1 are the x-coordinates of the two ends of the rod as shown in Fig. 3. If the length of the rod is l then

$$l = x_2 - x_1 \tag{1.16}$$

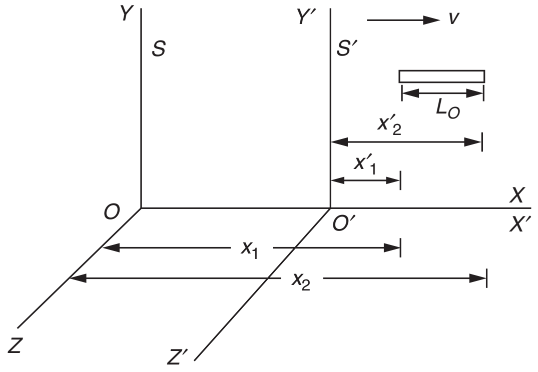


Figure 3 The length of the rod as measured by an observer.

Turning now to the Lorentz transformation equations, we see that

$$x_1' = \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad x_2' = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}}$$

Now

$$x_2' - x_1' = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}} - \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}}$$

Using Eq. (1.15), we get

$$l_0 = \frac{x_2 - vt}{\sqrt{1-(v^2/c^2)}} - \frac{x_1 - vt}{\sqrt{1-(v^2/c^2)}} = \frac{x_2 - x_1}{\sqrt{1-(v^2/c^2)}}$$

From Eq. (1.16), we get

$$l_0 = \frac{l}{\sqrt{1-(v^2/c^2)}}$$

Hence,

$$l = l_0\sqrt{1-(v^2/c^2)} \tag{1.17}$$

Thus the length of the rod as measured in the frame of reference S with respect to which the rod is moving is shorter by a factor of $\sqrt{1-(v^2/c^2)}$. A rod will be observed to have its maximum length when it is stationary.

This phenomenon is known as the *Lorentz–Fitzgerald length contraction*. It is not the consequence of some force 'squeezing' the rod, but it is a real physical phenomenon with observable physical effects. Note, however, that someone who actually looks at this rod as it passes by will not see a shorter rod. If the time that is required for the light from each point on the rod to reach the observer's eye is taken into account, the overall effect is that of making the rod appear as if it is rotated in space.

1.11.2 Time Dilation

A clock moving with a uniform velocity v relative to an observer at rest appears to him to go slow by a factor $\sqrt{1-(v^2/c^2)}$ *than when at rest. This effect is said to be time dilation*. In order to obtain a relation for time dilation, let a clock be placed at a point in the frame S and another at a point in the frame S' moving with velocity v with respect to the frame S along the positive x-axis. According to Lorentz transformation for time

$$t = \frac{t' + (x'v/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.18}$$

Again consider a light signal is emitted at point x' at time t_1' and another at the same location at time t_2' in the frame S' as measured by an observer in frame S. Therefore,

$$t_1 = \frac{t_1' + (x'v/c^2)}{\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad t_2 = \frac{t_2' + (x'v/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.19}$$

Therefore

$$\Delta t = t_2 - t_1 = \frac{t_2' + (x'v/c^2)}{\sqrt{1-(v^2/c^2)}} - \frac{t_1' + (x'v/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.20}$$

$$\Rightarrow \Delta t = t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-(v^2/c^2)}} \tag{1.21}$$

$$\Rightarrow \Delta t = \frac{\Delta t'}{\sqrt{1-(v^2/c^2)}} \tag{1.22}$$

Since, $v < c$, $\sqrt{1-(v^2/c^2)} < 1$. Therefore,

$$\Delta t > \Delta t' \tag{1.23}$$

In order to interpret this result, suppose $\Delta t'$ is the time interval between two 'ticks' of the clock. Then according to the clocks in S, these two 'ticks' are separated by a time interval Δt which, by Eq. (1.23) is greater than $\Delta t'$. Thus the time interval between 'ticks' is longer, as measured by the clocks in S, than what it is measured to be in S'. In other words, from the point of view of the frame of reference S, the clock (and all the clocks in S') is running slow. It appears from S that time is passing more slowly in S' than it is in S. This is the phenomenon of time dilation. *A clock will be observed to run at its fastest when it is in a stationary frame of reference. The clock is then measuring proper time.*

This phenomenon is just as real as length contraction. One of its best known consequences is that of the increase in the lifetime of a radioactive particle moving at a speed close to that of light. For example, it has been shown that if the lifetime of a species of radioactive particle is measured while stationary in a laboratory to be T', then the lifetime of an identical particle moving relative to the laboratory is found to be given by $T = T' / \sqrt{1-(v^2/c^2)}$, in agreement with Eq. (1.22).

1.11.3 Experimental Verification of Time Dilation (Example of Real Effect)

Time dilation has been verified by π mesons or μ mesons. These are created at the height of about 10 km in the earth atmosphere by the interaction of photons and are projected towards the earth surface with velocity 2.29×10^8 m/sec (or $v = 0.998c$). These mesons are unstable and decay with an average life-time of approximately 1.8×10^{-8} sec and 2.0×10^{-6} sec for π mesons and μ mesons, respectively. Hence, in this life-time μ mesons can travel a distance

$$d = \text{Velocity} \times \text{Decay time} = 2.29 \times 10^8 \times 2.0 \times 10^{-6} \approx 0.6 \text{ km}$$

This distance is small compared to the distance of 10 km required to reach the earth surface. This is possible because of time dilation effect. So, dilated life-time due to relativistic effect is

$$T = \frac{T'}{\sqrt{1-(v^2/c^2)}} = \frac{2.0 \times 10^{-6}}{\sqrt{1-\left(\dfrac{2.29 \times 10^8}{3.0 \times 10^8}\right)^2}} \cong 3.17 \times 10^{-5} \text{ sec}$$

In this dilated life-time μ mesons can travel a distance

$$d = \text{Velocity} \times \text{Dilated life-time} = 2.29 \times 10^8 \times 3.17 \times 10^{-5} \approx 9.5 \text{ km}$$

This explains the presence of μ mesons on the earth surface. Thus, this experiment is evidence for time dilation.

1.12 Twin Paradox in Special Relativity

Let us consider two twins A and B, each 30 years of age. Twin A remains at rest at the origin O and twin B takes a round trip space voyage to a star with velocity $v = 0.99c$ relative to A. The star is 20 light years away from O. We want to determine the age of A and B as B finishes his journey. According to A, the time taken by B in the round trip is

$$t_1 = \frac{30 \text{ light years}}{0.99c} = \frac{30c}{0.99c} = 30.3 \text{ years}$$

Thus, according to A his own age, as B completes the journey, is 30 + 30.3 = 60.3 years.

According to B, the time of journey (proper time interval) is as follows:

$$t_2 = 30 \times \sqrt{1-(v^2/c^2)} = 30\sqrt{1-(0.99)^2} = 4.2 \text{ years}$$

Hence, according to B, his own age after the journey is 30 + 4.2 = 34.2 years.

These two statements are different. After the space journey, will one of the two twins look younger than the other? This is twin paradox in the theory of special relativity.

1.13 Transformation of Velocities or Addition of Velocities

In order to obtain the transformation equations for velocities of a particle in the frame S and S', we consider the velocity in the frame S be u (u_x, u_y, u_z) and that in S' be u' (u_x', u_y', u_z'). Therefore,

$$u_x = \frac{dx}{dt} \quad \text{and} \quad u_x' = \frac{dx'}{dt'} \tag{1.24}$$

By Lorentz transformation equations

$$x' = \frac{x - vt}{\sqrt{1-(v^2/c^2)}}, \quad y = y', \quad z = z', \quad t' = \frac{t - (vx/c^2)}{\sqrt{1-(v^2/c^2)}}$$

Differentiating these equations, we get

$$dx' = \frac{dx - vdt}{\sqrt{1-(v^2/c^2)}}, \quad dy = dy', \quad dz = dz', \quad dt' = \frac{dt - (vdx/c^2)}{\sqrt{1-(v^2/c^2)}}$$

Dividing dx', dy' and dz' *by* dt' we get

$$\frac{dx'}{dt'} = \frac{dx - vdt}{dt - (vdx/c^2)} \Rightarrow u_x' = \frac{(dx/dt) - v}{1 - \frac{v}{c^2}\frac{dx}{dt}} \quad \Rightarrow u_x' = \frac{u_x - v}{1 - \frac{v\,u_x}{c^2}} \quad \left(\because \frac{dx}{dt} = u_x\right)$$

Similarly,

$$u_y' = \frac{dy'}{dt'} = \frac{dy\sqrt{1-(v^2/c^2)}}{dt - (vdx/c^2)} = \frac{\frac{dy}{dt}\sqrt{1-(v^2/c^2)}}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u_y\sqrt{1-(v^2/c^2)}}{1 - \frac{vu_x}{c^2}},$$

$$u_z' = \frac{dz'}{dt'} = \frac{dz\sqrt{1-(v^2/c^2)}}{dt - \frac{vdx}{c^2}} = \frac{\frac{dz}{dt}\sqrt{1-(v^2/c^2)}}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u_z\sqrt{1-(v^2/c^2)}}{1 - \frac{vu_x}{c^2}}$$

Thus,

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}; \quad u_y' = \frac{u_y\sqrt{1-(v^2/c^2)}}{1 - \frac{vu_x}{c^2}}; \quad u_z' = \frac{u_z\sqrt{1-(v^2/c^2)}}{1 - \frac{vu_x}{c^2}} \tag{1.25}$$

Equation (1.25) represents the *relativistic velocity transformation*. When the particle is moving along x-axis, then the component $u_y = u_z = 0$ and $u_x = u$. Hence,

$$u' = \frac{u - v}{1 - \frac{vu_x}{c^2}} \tag{1.26}$$

The inverse transformations are obtained by replacing v by $-v$ in Eq. (1.25). We have

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}}; \quad u_y = \frac{u'_y\sqrt{1-(v^2/c^2)}}{1 + \dfrac{vu'_x}{c^2}}; \quad u_z = \frac{u'_z\sqrt{1-(v^2/c^2)}}{1 - \dfrac{vu'_x}{c^2}} \tag{1.27}$$

In the non-relativistic case $(v/c) << 1$, and so Eq. (1.27) reduces to

$$u_x = u'_x + v; \quad u_y = u'_y; \quad u_z = u'_z \tag{1.28}$$

Equation (1.28) is equivalent to classical results which is same as the Galilean transformations. If the particle is photon moving with the velocity of light in the positive x-direction, that is $u'_x = c$, then its velocity observed by an observer in frame S is given by

$$u_x = \frac{u'_x + v}{1 + \dfrac{vu'_x}{c^2}} = \frac{c + v}{1 + \dfrac{vc}{c^2}} = \frac{c(c+v)}{(c+v)} = c$$

From the above equation we can conclude that the velocity transformations are consistent with the hypothesis of relativity, that is, no particle can move with the greater velocity than the velocity of light.

1.14 Variation of Mass with Velocity

Up to now, we have discussed that length and time are not absolute but depend upon the frame of reference in which they are measured. Newtonian mechanics suggests that the mass of the body remains constant; however, from the relativistic point of view under all circumstances, the mass of a body is not a consistent property, but it is the function of velocity of the body. Thus, the change of mass with velocity will obviously change the momentum, energy and force. The variation of mass with velocity is shown through the following relation:

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where m_0 the rest mass of the body and m is the moving body mass. In order to obtain the expression for the variation of mass with velocity, we consider two frames of reference S and S'. Frame S is at rest while S' is moving with velocity v along the x-direction. Suppose two bodies with masses m_1 and m_2 travel with the corresponding velocities u' and $-u'$ in the moving frame S' and u_1 and u_2 in the frame S (Fig. 4).

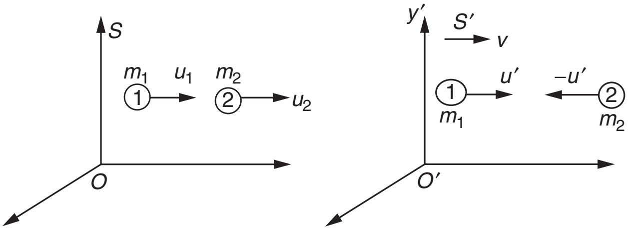


Figure 4 Variation of mass with velocity.

At the time of collision, the colliding particles are at rest with respect to the frame S' but move with velocity v relative to the frame S. Therefore, according to law of addition of relativistic velocities

$$u_1 = \frac{u' + v}{1 + \frac{vu'}{c^2}} \tag{1.29}$$

and

$$u_2 = \frac{-u' + v}{1 - \frac{vu'}{c^2}} \tag{1.30}$$

Using the principle of conservation of momentum, we can write

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v \tag{1.31}$$

Substituting the values of u_1 and u_2 from Eqs. (1.29) and (1.30) in Eq. (1.31), we get

$$m_1\left(\frac{u' + v}{1 + \frac{v\,u'}{c^2}}\right) + m_2\left(\frac{-u' + v}{1 - \frac{v\,u'}{c^2}}\right) = (m_1 + m_2)v \Rightarrow m_1\left(\frac{u' + v}{1 + \frac{v\,u'}{c^2}} - v\right) = m_2\left(v + \frac{u' - v}{1 - \frac{v\,u'}{c^2}}\right)$$

Simplifying and cancelling the common terms, the above equation reduces to

$$\frac{m_1}{m_2} = \frac{1 + \frac{vu'}{c^2}}{1 - \frac{vu'}{c^2}} \tag{1.32}$$

Using Eqs. (1.29) and (1.30), we find out the values of the following:

$$1 - \frac{u_1^{\,2}}{c^2} = 1 - \frac{\left(\frac{u' + v}{1 + \frac{v\,u'}{c^2}}\right)^2}{c^2} = 1 - \left(\frac{\frac{u' + v}{c}}{1 + \frac{v\,u'}{c^2}}\right)^2 = \frac{1 + \frac{v^2\,u'^2}{c^4} + \frac{2u'v}{c^2} - \frac{u'^2}{c^2} - \frac{v^2}{c^2} - \frac{2u'v}{c^2}}{\left(1 + \frac{v\,u'}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{u'^2}{c^2}\right) - \frac{v^2}{c^2}\left(1 - \frac{u'^2}{c^2}\right)}{\left(1 + \frac{v\,u'}{c^2}\right)^2} = \frac{\left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 + \frac{v\,u'}{c^2}\right)^2}$$

or

$$1 + \frac{v\,u'}{c^2} = \sqrt{\frac{\left(1 - \frac{u'^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{u_1^{\,2}}{c^2}}} \tag{1.33}$$

Similarly, we can obtain

$$1-\frac{vu'}{c^2}=\sqrt{\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}} \tag{1.34}$$

Substituting the values from Eqs. (1.34) and (1.33) in Eq. (1.32), we get

$$\frac{m_1}{m_2}=\frac{\sqrt{\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}}}{\sqrt{\frac{\left(1-\frac{u'^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}}}=\sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}}$$

If mass m_2 in the system S is at rest before collision, that is $u_2 = 0$, then

$$\frac{m_1}{m_2}=\sqrt{1-\frac{u_1^2}{c^2}} \Rightarrow m_1=\frac{m_2}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

If $m_1 = m$, $m_2 = m_0$ the rest mass of the body, and $u_1 = v$, then the above equation reduces to

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

This is the required formula for the variation of mass with velocity. The following are the important conclusions from the above equation:

1. As the velocity v of the particle relative to the observer increases, the mass of the particle increases.
2. If v tends to the velocity of light c, then m tends to infinity that means no material particle can have a velocity equal to or greater than the velocity of light.
3. If $v << c$, then v^2/c^2 is very small and may be neglected compared to 1, that is $m = m_0$.

1.15 Expression for the Relativistic Kinetic Energy

According to classical mechanics, force is defined as the rate of change of linear momentum. This is true for the relativistic mechanics as well. Thus, the force applied on a particle moving with relativistic velocity v is as follows:

$$F=\frac{dP}{dt}=\frac{d}{dt}(mv) \tag{1.35}$$

where *m* is the relativistic mass of the particle. Since *m* is the variable quantity, therefore

$$F = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{1.36}$$

If a force *F* displaces the particle through distance *ds*, then the work has done by the force increases the kinetic energy of the particle by dE_k. Therefore,

$$dE_k = F \cdot ds = \left[m\frac{dv}{dt} + v\frac{dm}{dt} \right] \cdot ds \tag{1.37}$$

or
$$dE_k = m\frac{dv}{dt}ds + v\frac{dm}{dt}ds$$

or
$$dE_k = m\frac{ds}{dt}dv + v\frac{ds}{dt}dm$$

or
$$dE_k = mvdv + v^2 dm \tag{1.38}$$

According to Einstein special theory of relativity,

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow m^2 = \frac{m_0^{\ 2}}{1-\frac{v^2}{c^2}} \quad \Rightarrow 1-\frac{v^2}{c^2}m^2 = m_0^{\ 2} \quad \Rightarrow (c^2 - v^2)m^2 = m_0^{\ 2}c^2$$

$$\Rightarrow (c^2m^2 - v^{\ 2}m^2) = m_0^{\ 2}c^2 \tag{1.39}$$

Differentiating Eq. (1.39), we get

$$(c^2\,2m\;dm - v^2\,2m\;dm - 2v\;dv\;m^2) = 0$$
$$\Rightarrow 2m\,[(c^2 - v^2)\;dm - mv\;dv] = 0$$
$$\Rightarrow (c^2 - v^2)\;dm = mv\;dv$$

Now substituting the value of *mv dv* in Eq. (1.38), we get

$$dE_k = (c^2 - v^2)\;dm + v^2 dm \quad \Rightarrow dE_k = c^2 dm \tag{1.40}$$

Integrating Eq. (1.40) between *m* and m_0 (when particle mass changes from rest mass m_0 to effective mass *m*), we have

$$E_k = \int_{m_0}^{m} dE_k = \int_{m_0}^{m} c^2 dm = c^2(m - m_0)$$

Since $m = \dfrac{m_0}{\sqrt{1-(v^2/c^2)}}$ therefore

$$E_k = c^2\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\right) = m_0c^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) \tag{1.41}$$

This is the required expression for kinetic energy of particle.

1.16 Einstein's Mass–Energy Relation

We know that the total energy of a moving particle is the sum of its kinetic energy and energy at rest, that is

$$E = E_k + E_0 \tag{1.42}$$

The rest energy of the particle is given by

$$E_0 = m_0c^2 \tag{1.43}$$

Now substituting the values of E_k and E_0 from Eqs. (1.41) and (1.43) in Eq. (1.42), we get

$$E = c^2\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\right) + m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

Since $m = \dfrac{m_0}{\sqrt{1-(v^2/c^2)}}$, therefore

$$E = mc^2$$

This is well known *Einstein mass–energy relation*. The relativistic kinetic energy can be expressed as

$$E_k = (m-m_0)c^2 = c^2\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} - m_0\right) = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2$$

Using binomial theorem we have

$$E_k = m_0c^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - m_0c^2 = m_0c^2\left[1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right] - m_0c^2$$

Since $v^2/c^2 << 1$, we have

$$E_k = m_0c^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - m_0c^2 = \frac{1}{2}m_0v^2$$

This relation is the classical result for the kinetic energy.

1.17 Relativistic Relation between Energy and Momentum

We know that total energy of a moving particle is

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} \tag{1.44}$$

where m_0 is the rest mass of the particle. Since $p = mv$ we have $v = p/m$. Substituting the value of v in Eq. (1.44) we get

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2}{m^2 c^2}}} = \frac{m_0 c^2}{\sqrt{1-\frac{p^2 c^2}{m^2 c^4}}}$$

$$\Rightarrow E = \frac{m_0 c^2}{\sqrt{1-\frac{p^2 c^2}{(mc^2)^2}}} = \frac{m_0 c^2}{\sqrt{1-\frac{p^2 c^2}{E^2}}} \quad [\because E = mc^2] \tag{1.45}$$

Squaring Eq. (1.45), we get

$$E^2 = \frac{m_0^{\,2} c^4}{1-\frac{p^2 c^2}{E^2}} \Rightarrow E^2\left(1-\frac{p^2 c^2}{E^2}\right) = m_0^{\,2} c^4$$

$$\Rightarrow E^2 - p^2 c^2 = m_0^{\,2} c^4 \tag{1.46}$$

This is the required expression.

1.18 Massless Particles

A particle which has zero rest mass is known as massless particle. According to the relativistic relation between energy and momentum

$$E^2 - p^2 c^2 = m_0^{\,2} c^4$$

For massless particle, $m_0 = 0$. Therefore

$$E = pc \quad \text{or} \quad p = \frac{E}{c}$$

Thus, we can say that massless particle has energy pc and momentum E/c and moves with the velocity of light.

Solved Examples

Example 1

Prove that $x^2 + y^2 + z^2 - c^2 t^2$ is invariant under Lorentz transformation.

or

Show that space–time interval between two events remains invariant under Lorentz transformation.

Solution: The Lorentz transformation equations are

$$x = \frac{x' + v\,t'}{\sqrt{1-(v^2/c^2)}}; \; y = y'; \quad z = z'; \quad t = \frac{t' + \frac{xv}{c^2}}{\sqrt{1-(v^2/c^2)}}$$

Now

$$
\begin{aligned}
x^2 + y^2 + z^2 - c^2t^2 &= \left(\frac{x' + v\,t'}{\sqrt{1-(v^2/c^2)}}\right)^2 + y'^2 + z'^2 - c^2\left(\frac{t' + \dfrac{xv}{c^2}}{\sqrt{1-(v^2/c^2)}}\right)^2 \\
&= \frac{1}{1-(v^2/c^2)}\left[x'^2 + v^2\,t'^2 + 2\,t'vx' - c^2t'^2 - \frac{v^2x'^2}{c^2} - 2\,t'vx'\right] + y'^2 + z'^2 \\
&= \frac{1}{1-(v^2/c^2)}\left[x'^2 + v^2\,t'^2 - c^2t'^2 - \frac{v^2x'^2}{c^2}\right] + y'^2 + z'^2 \\
&= \frac{1}{1-(v^2/c^2)}\left[x'^2\left(1-\frac{v^2}{c^2}\right) + v^2\,t'^2 - c^2t'^2\right] + y'^2 + z'^2 \\
&= \frac{1}{1-(v^2/c^2)}\left[x'^2\left(1-\frac{v^2}{c^2}\right) - c^2t'^2\left(1-\frac{v^2}{c^2}\right)\right] + y'^2 + z'^2 \\
&= [x'^2 - c^2t'^2] + y'^2 + z'^2 \\
&= x'^2 + y'^2 + z'^2 - c^2t'^2
\end{aligned}
$$

Hence, $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformation.

Example 2

Determine the time taken by the rocket to reach a distant star and return to earth with a constant velocity $v = \sqrt{0.99}\,c$, if the distance to the star is 6 light years.

Solution: Here, we have

$$x = 2 \times 6 \text{ light years} = 2 \times 6 \times 365 \times 24 \times 60 \times 60 \times c = 3.81 \times 10^8 c \text{ and } v = \sqrt{0.99}\,c$$

Now

$$t' = \frac{t - (xv/c^2)}{\sqrt{1-(v^2/c^2)}} \tag{1.47}$$

where t is the time taken by the rocket to reach the star and back to the earth as measured by an observer in a stationary frame at earth and t' is the time measured by the clock in the moving rocket for the total journey. Since $t = x/v$, therefore

$$t' = \frac{\dfrac{x}{v} - \dfrac{xv}{c^2}}{\sqrt{1-(v^2/c^2)}}$$

$$\Rightarrow t' = \frac{x}{v}\sqrt{1-(v^2/c^2)} \tag{1.48}$$

Substituting the values of x and v in Eq. (1.48), we get

$$t' = \frac{3.81 \times 10^8 c}{\sqrt{0.99}\ c}\sqrt{1-(0.99c^2/c^2)} = 3.81 \times 10^6 \text{ sec}$$

Example 3

In an inertial frame S, two lights (red and blue) are separated by a distance $\Delta x = 2.45$ km, with the red light at the larger value of x. The blue light flashes and 5.35 microsecond later the red light flashes. Frame S' is moving in a direction of increasing x with speed of $v = 0.85c$. What is the distance between the two flashes and the time between them as measured in S'?

Solution: According to Lorentz transformation equations

$$x_1' = \frac{x_1 - v\,t_1}{\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad x_2' = \frac{x_2 - vt_2}{\sqrt{1-(v^2/c^2)}}$$

$$t_1' = \frac{t_1 - (x_1 v/c^2)}{\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad t_2' = \frac{t_2 - (x_2 v/c^2)}{\sqrt{1-(v^2/c^2)}}$$

The distance between the red light and blue light flashes with respect to an observer in moving frame S' is

$$\Delta x' = x_2{}' - x_1{}' = \frac{(x_2 - x_1) - v\,(t_2 - t_1)}{\sqrt{1-(v^2/c^2)}}$$

Here, $\Delta x = 2.45$ km $= 2.45 \times 10^3$ m, $\Delta\,t = 5.35$ microsecond $= 5.35 \times 10^{-6}$ sec and $c = 3 \times 10^8$ m. After substituting these values in the above equation, we get

$$\Delta x' = 20.78 \times 10^2 \text{ m or } 2.08 \text{ km}$$

Similarly,

$$\Delta\,t' = t_2' - t_1' = \frac{(t_2 - t_1) - v\left(\dfrac{x_2 - x_1}{c^2}\right)}{\sqrt{1-(v^2/c^2)}} = \frac{\Delta\,t - \dfrac{v\,\Delta x}{c^2}}{\sqrt{1-(v^2/c^2)}}$$

Since $v = 0.85c$, we have

$$\Delta\,t' = -1.35 \times 10^{-5} \text{ sec or } -0.135 \text{ microsecond}$$

Negative sign shows that red light flash comes before the blue one.

Example 4

How fast would a rocket have to go relative to an observer for its length to be contracted to 95% of its length at rest?

Solution: We have to find v. We have

$$l = 95\% \text{ of } l_0 = 0.95\ l_0 \quad \text{or} \quad \frac{l}{l_0} = 0.95$$

Therefore,

$$l = l_0\,\sqrt{1-(v^2/c^2)} \Rightarrow \frac{v^2}{c^2} = 1 - \left(\frac{l}{l_0}\right)^2 = 1 - (0.95)^2 \Rightarrow \frac{v}{c} = 0.0975 \Rightarrow v = 0.0975\,c$$

Example 5

Determine the length and orientation of a rod of length 10 m in a frame of reference which is moving with 0.6c velocity in a direction making 30° angle with the rod.

Solution: The proper length of the rod along the direction of the moving frame is $l_0 = 10\cos 30°$. Therefore, the measured length in the moving frame is

$$l_x = l_0\sqrt{1-(v^2/c^2)} = 10\cos 30°\sqrt{1-[(0.6c)^2/c^2]} = 8\cos 30° = 4\sqrt{3}\ \text{m}$$

Since, the length does not contract perpendicular to the direction of the motion of moving frame, we have

$$l_y = 10\sin 30° = 5\ \text{m}$$

Hence, the length of the rod observed in the moving frame is given by

$$l = \sqrt{l_x^{\,2} + l_y^{\,2}} = \sqrt{48+25} = \sqrt{73} = 8.54\ \text{m}$$

If the rod makes an angle θl with x-axis of the moving frame, then

$$\tan\theta l = \frac{l_y}{l_x} = \frac{5}{4\sqrt{3}} = 0.72 \quad \text{or} \quad \theta l = 35.8°$$

Example 6

Show that if V_0 is the rest volume of a cube of side l_0, then $V_0(1-\beta^2)^{1/2}$ is the volume viewed from a reference frame moving with uniform velocity v in a direction parallel to an edge of the cube.

or

A cube with its sides of proper length l_o is viewed from a reference frame moving with uniform velocity v, parallel to an edge of the cube. Deduce the expression for the volume of the cube for the observer.

Solution: Since one side of a cube is l_0, therefore the rest volume of the cube for an observer with respect to the cube $= l_o \times l_o \times l_o = l_o^3$. Now the edge along x-axis as observed by an observer in moving frame will get contracted to $l_x = l_o\sqrt{1-(v^2/c^2)}$. But no contraction will be observed in the length of the other two edges. Therefore, the volume observed by the observer is

$$V = l_x\, l_y\, l_z = l_o\sqrt{1-\frac{v^2}{c^2}}\ l_o \cdot l_o = l_o^{\,3}\sqrt{1-\frac{v^2}{c^2}}$$

Since $v^2/c^2 = \beta^2$, therefore

$$V = V_0\sqrt{1-\beta^2} \quad [\because V_0 = l_0^{\,3}]$$

Example 7

A rod placed in a frame of reference is moving with a velocity $0.6c$ in a direction parallel to its length. Calculate the percentage of contraction.

Solution: We know that

$$l = l_0 \sqrt{1-(v^2/c^2)}$$

Given that $v = 0.6c$. Therefore

$$l = l_0 \sqrt{1-[(0.6c)^2/c^2]} = l_0 \sqrt{1-0.36} = l_0 \sqrt{0.64} = 0.8\, l_0$$

Percentage contraction is given by

$$\text{Percentage contraction} = \frac{l_0 - 0.8\, l_0}{l_0} \times 100 = 20\%$$

Example 8

A meson has a speed $0.8c$ relative to the ground. Find how far the meson travels relative to the ground if its speed remains constant and the time of its flight, relative to the system, in which it is at rest, is 2×10^{-8} sec. Also calculate the distance travelled by the meson in absence of relativity effect.

or

The proper mean life of a meson is 2×10^{-8} sec. Calculate the mean life of a meson moving with a velocity $0.8c$.

Solution: Time of flight of meson relative to the earth is

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v^2/c^2)}}$$

We have

$$\text{Proper mean life time of meson } \Delta t' = 2 \times 10^{-8} \text{ sec}$$

Therefore

$$\Delta t = \frac{2 \times 10^{-8}}{\sqrt{1-\dfrac{(0.8\,c)^2}{c^2}}} = \frac{2 \times 10^{-8}}{\sqrt{1-0.64}} = \frac{2 \times 10^{-8}}{0.6} = 3.33 \times 10^{-8} \text{ sec}$$

Distance travelled by the mason relative to the ground is given by

$$\text{Distance travelled } = v\, \Delta t = 0.8\, c \times 3.33 \times 10^{-8} = 2.66 \times 10^{-8} c \text{ or } 7.98 \text{ m}$$

Distance travelled by the meson in absence of relativity effect

$$v\, \Delta t = 0.8c \times 2 \times 10^{-8} = 1.6 \times 10^{-8} c$$

Example 9

The proper life of meson is 2×10^{-8} sec. If a beam of these mesons of velocity $0.8c$ is produced, calculate the distance the beam can travel before the flux of the meson beam is reduced to $1/e^2$ times the initial flux.

Solution: According to the time dilation formula

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v^2/c^2)}}$$

Proper mean life time of meson $\Delta t' = 2\times10^{-8}$ sec. Therefore

$$\Delta t = \frac{2\times10^{-8}}{\sqrt{1-\dfrac{(0.8c)^2}{c^2}}} = \frac{2\times10^{-8}}{\sqrt{1-0.64}} = \frac{2\times10^{-8}}{0.6} = 3.33\times10^{-8}\ \text{sec}$$

The distance travelled by the meson beam before the flux reduced to $1/e^2$ times the initial flux is

$$v\,2\,\Delta t = 0.8c\times2\times3.33\times10^{-8} = 7.98\times10^{-8}\,c$$

Example 10

A clock keeps correct time. With what speed should it be moved relative to an observer so that it may seem to lose 2 minutes in 24 hours.

Solution: Suppose the clock is at rest in frame S. In this frame the clock should show $\Delta t' = 23$ h 58 min $= 1438$ min. The observer relative to which frame S is moving with speed v will measure the time $\Delta t = 24$ h $= 1440$ min in his clock. Then according to the time dilation formula

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v^2/c^2)}} \Rightarrow 1440 = \frac{1438}{\sqrt{1-(v^2/c^2)}} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1438}{1440}$$

$$\Rightarrow 1-\frac{v^2}{c^2} = \left(\frac{1438}{1440}\right)^2 \Rightarrow \frac{v^2}{c^2} = 1-0.9986 = 0.0014 \Rightarrow v = 0.0374\,c \quad \text{or} \quad 1.12\times10^7\ \text{m/sec}$$

Example 11

Prove that no signal can travel faster than the velocity of light.

Solution: Put $u_x' = c$ and $v = c$ in following relation:

$$u_x = \frac{u_x' + v}{1+\dfrac{vu_x'}{c^2}}$$

We get

$$u_x = \frac{c+c}{1+\frac{cc}{c^2}} = \frac{2c}{2} = c$$

This means that the velocity of light in vacuum is the maximum velocity in nature and no signal can travel faster than light in vacuum.

Example 12

Prove that no velocity transformations are consistent with the hypothesis of relativity.

Solution: Put $u_x' = c$ in the following relation:

$$u_x = \frac{u_x' + v}{1+\frac{v\,u_x'}{c^2}}$$

We get

$$u_x = \frac{u_x' + v}{1+\frac{v\,u_x'}{c^2}} = \frac{c+v}{1+\frac{v\,c}{c^2}} = \frac{c(c+v)}{(c+v)} = c$$

Example 13

Show that a particle which travels with the speed of light must have a zero rest mass.

Solution: We know that the relativistic mass of a particle with speed v is

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where m_0 is the rest mass. If $v = c$, then

$$m = \frac{m_0}{\sqrt{1-1}} \Rightarrow m_0 = 0$$

Therefore, a particle travelling with the speed of light must have a zero rest mass.

Example 14

Calculate the velocity of a particle at which its mass will be 3 times of its rest mass.

Solution: We know that

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here $m = 3m_0$. So

$$3m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{9} \quad \Rightarrow v^2 = \frac{8}{9}c^2 \quad \Rightarrow v = \frac{2\sqrt{2}}{3}c \quad \Rightarrow v = 0.88 \text{ c}$$

Example 15

A particle is moving with 30% of the velocity of light. Compare its relativistic mass with its rest mass.

Solution: We know that

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

We have

$$\frac{v}{c} = \frac{30}{100} = 0.3$$

So

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-(0.3)^2}} = 0.11$$

Hence, the relativistic mass is 2.3 times the rest mass of the particle.

Example 16

Prove that $\frac{1}{2}mv^2$, where $m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$, is not equal to the kinetic energy of a particle moving at relativistic velocity.

Solution: The relativistic kinetic energy can be expressed as

$$E_k = (m-m_0)c^2 = c^2\left(\frac{m_0}{\sqrt{1-(v^2/c^2)}} - m_0\right) = \frac{m_0c^2}{\sqrt{1-(v^2/c^2)}} - m_0c^2$$

Using binomial theorem

$$E_k = m_0c^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - m_0c^2 = m_0c^2\left[1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\cdots\right] - m_0c^2$$

Since $(v^2/c^2) << 1$, so

$$E_k = m_0c^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - m_0c^2 = \frac{1}{2}m_0v^2$$

Since, $m \neq m_0$, therefore, $\frac{1}{2}mv^2$, is not equal to the kinetic energy of a particle moving at relativistic velocity.

Example 17

Show that the momentum of a particle of rest mass m_0 and kinetic energy E_k is given by the expression

$$p = \sqrt{\frac{E_k^{\,2}}{c^2} + 2m_0 E_k}.$$

Solution: We know that

$$E_k = (m - m_0)c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0 c^2 = E - m_0 c^2$$

where $E = \sqrt{m_0^{\,2} c^4 + p^2 c^2}$. Now

$$E_k = \sqrt{m_0^{\,2} c^4 + p^2 c^2} - m_0 c^2 \Rightarrow E_k + m_0 c^2 = \sqrt{m_0^{\,2} c^4 + p^2 c^2}$$

Squaring both sides and deleting common terms, we get

$$p^2 = \frac{E_k^{\,2}}{c^2} + 2m_0 E_k \Rightarrow p = \sqrt{\frac{E_k^{\,2}}{c^2} + 2m_0 E_k}$$

Example 18

The total energy of a particle is exactly thrice its rest energy. Find its speed.

Solution: The total energy of a particle is given by

$$E = (m - m_0)c^2 + m_0 c^2$$

Given that energy of a particle is exactly thrice its rest energy, so $E = 3\,m_0 c^2$. Hence we have

$$3\,m_0 c^2 = (m - m_0)c^2 + m_0 c^2 = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow 1 - \frac{v^2}{c^2} = \frac{1}{9} \Rightarrow v = \frac{2\sqrt{2}}{3}c$$

Example 19

Show that the rest mass of a photon is zero.

Solution: We know that a photon travels with the velocity of light and its momentum is given by the following relation:

$$p = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow m_0 = \frac{p\sqrt{1 - \frac{v^2}{c^2}}}{v}$$

Now for a photon $v = c$. Therefore we have

$$m_0 = \frac{p\sqrt{1 - \frac{c^2}{c^2}}}{v} = 0$$

which implies that the rest mass of a photon is zero.

Short Answers of Some Important Questions

1. What is the role of relativity?

Answer: The role of relativity appears to be that of specifying the properties of space and time, the arena in which all physical processes take place.

2. Distinguish between inertial and non-inertial frames of reference.

Answer: In inertial frames of reference, bodies obey Newton's law of inertia and other laws of Newtonian or classical mechanics. In such a frame of reference, if no net external force acts upon a body, the body will move with zero acceleration, that is, moving with a constant velocity. So it is also known as unaccelerated frames of reference. On the other hand, the frames of reference in which Newton's law is not valid is said to be non-inertial. In other words, the accelerated frames of reference are called non-inertial.

3. Write down the postulates of special theory of relativity.

Answer: There are two postulates of special theory of relativity

1. All the laws of physics are the same in every inertial frame of reference.
2. The speed of light is independent of the motion of its source.

4. What do you understand by proper length?

Answer: The length l_0 of the rod measured by an observer in the frame in which the rod is at rest is called proper length or actual length.

5. What is proper time?

Answer: A clock will be observed to run at its fastest when it is stationary in a frame of reference. The clock is then said to be measuring proper time.

6. What is massless particle?

Answer: A particle which has zero rest mass is known as massless particle. The massless particle has energy *pc* and momentum *E/c* and moves with the velocity of light. For example, photon is a massless particle.

7. Does light comprising photons have mass? If no, then how do photons have momentum?

Answer: Since light is composed of photons and photon is a massless particle, therefore, the photon has energy and momentum but no mass. According to the relation $E^2 - p^2c^2 = m_0{}^2c^4$, we can prove that the energy and momentum of the photon are $E = p\,c$ and $p = E / c$, respectively, due to zero rest mass of photon.

8. Why is $E_0 = m_0c^2$ called the rest mass of energy of an object?

Answer: We know that the total energy of an object is $E = \dfrac{m_0c^2}{\sqrt{1-(v^2/c^2)}}$. When the object is at rest, that is the velocity is zero, then energy will be m_0c^2 which is denoted by E_0. This energy is known as rest mass energy.

Important Points and Formulas

1. According to special theory of relativity, everything in the universe is relative, nothing is absolute, all rest and motions are relative, position and time are relative, etc.
2. A system of coordinate axes which defines the position of a particle or specifies the location of an event is called a frame of reference.
3. The main objective of Michelson–Morley experiment was to confirm the existence of ether.
4. The fringe shift in Michelson–Morley experiment is

$$\Delta n = \frac{2L'v^2}{c^2}\cdot\frac{1}{\lambda} = 0.4$$

5. Lorentz transformation equations for space and time are

$$x' = \frac{x - v\,t}{\sqrt{1-(v^2/c^2)}};\quad y = y';\ z = z' \text{ and}$$

$$t' = \frac{t - \dfrac{xv}{c^2}}{\sqrt{1-(v^2/c^2)}}$$

6. Inverse Lorentz transformation equations are

$$x = \frac{x' + v\,t'}{\sqrt{1-(v^2/c^2)}},\ y = y',\ z = z' \text{ and}$$

$$t = \frac{t' + \dfrac{xv}{c^2}}{\sqrt{1-(v^2/c^2)}}$$

7. Length contraction equation

$$l = l_0\sqrt{1-(v^2/c^2)}$$

8. Time dilation equation

$$\Delta t = \frac{\Delta t'}{\sqrt{1-(v^2/c^2)}}$$

9. Relativistic velocity transformation equations

$$u'_x = \frac{u_x - v}{1 - \dfrac{v\,u_x}{c^2}},\quad u'_y = \frac{u_y\sqrt{1-(v^2/c^2)}}{1 - \dfrac{vu_x}{c^2}},$$

$$u'_z = \frac{u_z\sqrt{1-(v^2/c^2)}}{1 - \dfrac{vu_x}{c^2}}$$

10. Variation of mass with velocity

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

11. Einstein's mass–energy relation: $E = mc^2$.
12. The relativistic kinetic energy

$$E_k = (m - m_0)c^2 = c^2\left(\frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} - m_0\right)$$

$$= \frac{m_0c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} - m_0c^2$$

13. Energy momentum relation

$$E^2 - p^2c^2 = m_0{}^2c^4$$

14. A particle which has zero rest mass is known as massless particle.

Multiple Choice Questions

1. Relativistic mechanics is applicable
(a) for low velocity
(b) for velocity close to the velocity of light
(c) for both low and high velocity
(d) None of above

2. Galilean transformations are applicable
(a) for relativistic motion
(b) for non-relativistic motion
(c) For both (a) and (b)
(d) Neither for (a) nor (b)

3. Which of the following is an invariant under Galilean transformations?
(a) Velocity **(b)** Momentum
(c) Acceleration **(d)** Charge

4. Lorentz transformations are applicable
(a) for relativistic motion
(b) for non-relativistic motion
(c) For both (a) and (b)
(d) None of these

5. An inertial frame of reference is one which
(a) does not accelerate
(b) remains at absolute rest
(c) remains at absolute motion
(d) attached to an observer

6. Michelson–Morley experiment showed that
(a) there is no absolute frame; all the frames are relative
(b) there is absolute frame; all the frames are relative
(c) relativistic mechanics is correct for high velocities
(d) relativistic mechanics is correct for low velocities

7. The rest mass of photon is
(a) infinity **(b)** zero
(c) E/c^2 **(d)** Ec^2

8. The presence of μ mesons in the atmosphere of earth provided the experimental verification of
(a) length contraction
(b) time dilation
(c) Galilean transformation
(d) Michelson–Morley experiment

9. The expression of force for relativistic motion is
(a) $F = m_0 \dfrac{dv}{dt}\left(1 - \dfrac{v^2}{c^2}\right)^{1/2}$
(b) $F = m_0 \dfrac{dv}{dt}\left(1 - \dfrac{v^2}{c^2}\right)^{3/2}$
(c) $F = m_0 \dfrac{dv}{dt}\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$
(d) $F = m_0 \dfrac{dv}{dt}\left(1 - \dfrac{v^2}{c^2}\right)^{-3/2}$

10. Which of the following is an example of mass–energy relation?
(a) Pair production
(b) Presence of μ mesons in earth atmosphere
(c) Both (a) and (b)
(d) None of these

11. The relativistic velocity addition theorem is consistent with
(a) first postulate of special theory of relativity
(b) second postulate of special theory of relativity
(c) Einstein's mass–energy relation
(d) None of these

12. Ether concept was discarded after
(a) Michelson–Morley experiment
(b) the first postulate of special theory of relativity
(c) the second postulate of special theory of relativity
(d) None of these

13. Two photons are approaching each other. The speed of one photon observed by the other will be
(a) zero **(b)** infinity
(c) c **(d)** None of these

14. The velocity of massless particle is
(a) same as that of light in free space
(b) less than that of light in free space
(c) more than that of light in free space
(d) independent of light in free space

15. Massless particles are

(a) electrons **(b)** photons

(c) gravitons **(d)** neutrons

16. Two particles came toward each other with speed 0.7*c* with respect to laboratory. What is their relative speed?

(a) 0.939*c* **(b)** 1.41*c*

(c) 0.838*c* **(d)** 0.56*c*

17. At what speed will the mass of a body be 2.25 times its rest mass?

(a) 1.68×10^8 m/sec **(b)** 2.68×10^8 m/sec

(c) 0.983×10^8 m/sec **(d)** None of these

18. The rest energy of a particle is

(a) mc^2 **(b)** E

(c) m_0c^2 **(d)** $(m - m_0)c^2$

19. The kinetic energy of a particle in relativistic mechanics is

(a) $(m - m_0)c^2$ **(b)** mc^2

(c) m_0c^2 **(d)** None of these

20. The speed of a 0.1 meV electrons according to relativistic mechanics is

(a) 2.48×10^8 m/sec **(b)** 1.64×10^8 m/sec

(c) 0 **(d)** *c*

Short Answer Type Questions

1. What is principle of relativity?
2. What do you understand by frame of reference?
3. Define inertial and non-inertial frame of reference with suitable example.
4. How does the principle of relativity lead the constancy of speed of light in all inertial frames?
5. Explain whether earth is inertial or non-inertial frame of reference.
6. Write Galilean transformations for space and time.
7. Why was it assumed that the space is filled with hypothetical medium ether?
8. What is the objective of Michelson–Morley experiment?
9. Discuss the importance of negative results of Michelson–Morley experiment.
10. State the postulate of special theory of relativity.
11. Write down Lorentz and inverse Lorentz transformation equations.
12. What is length contraction?
13. What do you mean by mass energy equivalence relation?
14. What do you understand by rest mass?

Long Answer Type Questions

1. What is principle of relativity? Discuss the invariance of speed of light.
2. What is frame of reference? Define inertial and non-inertial frame of reference with suitable example.
3. What are Galilean transformations for space and time?
4. What is objective of Michelson–Morley experiment? Why was Michelson–Morley experiment performed?
5. State the postulate of special theory of relativity and derive Lorentz transformation.
6. Prove that $E^2 - p^2c^2$ is invariant under Lorentz transformation.
7. Prove that no signal can travel faster than the velocity of light.
8. What is meant by relativistic length contraction? What is proper length?
9. What do you meant by time-dilation? What is proper time?
10. Derive the expression for relativistic law of addition of velocities. Hence show that *c* is the ultimate speed. Prove that the law is in conformity with the principle of constancy of speed of light.

Numerical Problems

1. Determine the time taken by the rocket to reach a distant star and return to earth with a constant velocity $v = 0.8\,c$ if the distance to the star is 3 light years.
2. An event occurs at $x = 100$ km, $y = 10$ km and $z = 1.0$ km at $t = 2.0 \times 10^{-4}$ sec in a reference frame S. Another frame S' is moving with speed 0.95 relative to S along the common x–x' axis, the origins coinciding at $t = t' = 0$. Compute the coordinates x', y', z', t' of events in S'.
3. A projectile is projected with a velocity v at an angle θ from the horizontal in a frame S. Show that in another frame S' moving with velocity $v \cos \theta$ relative to S along the positive x-direction, the projectile will appear to have only vertical motion.
4. A spacing 50 m long passes the earth at a speed of 2.8×10^{8} m/sec. What will be its apparent length?
5. How fast a rocket should be relative to an observer for its length to be contracted to 80% of its length at rest?
6. Determine the length and orientation of a rod of length 20 m in a frame of reference which is moving with $0.6c$ velocity in a direction making 45° angle with the rod.
7. A rod placed in a frame of reference is moving with a velocity $0.8\,c$ in a direction parallel to its length. Calculate the percentage of contraction.
8. A meson has a speed $0.6c$ relative to the ground. Find how far the meson travels relative to the ground if its speed remains constant and the time of its flight, relative to the system, in which it is at rest, is 2.5×10^{-8} sec. Also calculate the distance travelled by the meson in absence of relativity effect.
9. The proper life of meson is 2.5×10^{-8} sec. If a beam of these mesons of velocity $0.8c$ is produced, calculate the distance the beam can travel before the flux of the meson beam is reduced to $1/e^{4}$ times the initial flux.
10. A clock keeps correct time. With what speed should it be moved relative to an observer so that it may seem to lose 4 minutes in 24 hours.
11. A clock keeps correct time. With what speed should it be moved relative to an observer so that it may seem to lose 4 minutes in 36 hours.
12. Calculate the velocity of a particle at which its mass will be 5 times of its rest mass.
13. The mass of moving electron is 11 times its rest mass. Find its kinetic energy and momentum.

Answers

Multiple Choice Questions

1. (b)
2. (b)
3. (c)
4. (a)
5. (a)
6. (a)
7. (b)
8. (b)
9. (d)
10. (a)
11. (b)
12. (a)
13. (c)
14. (a)
15. (c)
16. (a)
17. (b)
18. (c)
19. (a)
20. (b)

Numerical Problems

1. 12.5 year
2. $x' = 137.8$ km, $y' = 10$ km, $z' = 1.0$ km, $t' = -3.74 \times 10^{-4}$
4. 18 m
5. $0.45c$
6. 17.26 m, $\theta = \tan^{-1}(10/7)$
7. 40%
8. 5.6 m, 4.5 m
9. 40 m
10. $0.0744c$
11. $0.0608c$
12. $0.9798c$
13. 5.12 MeV, 2.99×10^{-21} kg-m/sec

2 Wave Mechanics

LEARNING OBJECTIVES

After completing this chapter, you will be able to understand:

- Wave-particle duality.
- de-Broglie matter waves.
- Phase and group velocities.
- Davisson–Germer experiment.
- Heisenberg uncertainty principle and its applications.
- Wave function and its significance.
- Schrödinger's wave equation – particle in one-dimensional potential box.
- Eigenvalues and eigenfunctions.

2.1 Introduction

In the middle of the 19th and 20th century, a revolution in physics took place in explaining the particles of atomic dimensions. Classical mechanics could not explain the behavior of systems, such as atoms, molecules and nuclei. This revolutionary approach for atomic dimension systems is known as quantum mechanics or wave mechanics. Before going to quantum mechanical approach, one has to understand the shortcomings of classical mechanics. These are as follows:

1. It could not explain the stability of atoms.
2. It does not hold in the region of atomic dimensions.
3. It could not explain the observed spectrum of blackbody radiation.
4. It could not explain the observed variation of specific heat of metals and gases.
5. It could not explain the origin of discrete spectra of atoms.

Apart with the above, classical mechanics also could not explain a large number of observed phenomenon such as photoelectric effect, Compton effect, Raman effect, etc. Though there are many factors responsible for the failure of classical mechanics to explain the various phenomena, the most significant is the consideration of continuous distribution of energy. In fact, at the microscopic level (in the region of atomic or subatomic level) the concept for interpretation of energy distribution is discrete rather than continuous.

This chapter is simply an introduction to understand the basic concepts of quantum mechanics such as wave-particle duality, de-Broglie waves and Heisenberg principle. Wave function and Schrödinger wave equation are also a part of this chapter.

2.2 Wave-Particle Duality

Light, the electromagnetic radiation, possesses dual character of a particle as well as a wave. The wave-particle duality is only the concept of energy transmission of radiation. When we talk about particle concept, it is easy to understand if it is described by its properties like mass, velocity, momentum, energy and a definite position in space. The phenomena like photoelectric effect, Compton effect take place due to interaction of radiation as particle with matter which evidently has particle properties. Planck's quantum theory quanta successfully explain these phenomenon. On the other hand, wave concept is comparatively difficult to understand. Unlike particle, a wave is like a disturbance which spreads out over a large region. It cannot be located here or there. It is specified by its wavelength, frequency, amplitude, energy and momentum. Various optical phenomenon such as interference, diffraction and polarization strongly validate the wave character of electromagnetic radiation. ***Hence from the above facts it can be said that radiation sometimes behaves as particle nature and sometimes as wave and both cannot be separated***. It can also be understood that both particle and wave properties of radiation cannot appear simultaneously.

2.3 de-Broglie Hypothesis

In 1924, de-Broglie proposed that wave particle duality is not only associated with radiation, but matter such as electron, proton, neutron, etc. also possess this dualism characteristic. Therefore, a wave is always associated with the moving particle whether it is matter or radiation and controls the particle. He suggested that if electromagnetic radiation can act as a particle at some time and a wave at other time, then matter should also behave as particle at some time and wave at other. However, there was no experimental proof at that time and the hypothesis was totally theoretical and was simply based on the fact that nature loves symmetry. The occurrence of properties of wave or particle depends upon the conditions under which the particular phenomenon takes place. But remember both particle and wave cannot appear together.

According to de-Broglie hypothesis, a *particle in motion always has a wave associated with it and the motion of the particle is guided by that wave. These waves are called de-Broglie waves or matter waves.* The wavelength λ of matter waves associated with a moving particle is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mK}} \tag{2.1}$$

where p is the momentum, m is the mass, v is velocity and K is the kinetic energy of the particle. This relation is true when $v << c$. However, when v is comparable to c then relativistic effects come into the picture and hence we cannot take m as the rest mass of the particle, but it will be the relativistic mass and will be given as

$$m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$$

So, Eq. (2.1) will take the form

$$\lambda = \frac{h}{p} = \frac{h\sqrt{1-(v^2/c^2)}}{m_0 v} \tag{2.2}$$

where m_0 is the rest mass and c is the velocity of light.

2.4 de-Broglie's Wavelength

According to Planck's hypothesis, the energy of a photon is given by

$$E = h\nu \tag{2.3}$$

where h is the Planck's constant and ν is the frequency. If mass of the particle is converted to energy, then energy is given by

$$E = m\,c^2 \tag{2.4}$$

where m is the mass of photon. From Eqs. (2.3) and (2.4), we have

$$h\,\nu = m\,c^2 \Rightarrow h\,\frac{c}{\lambda} = m\,c^2 \quad \left(\because \nu = \frac{c}{\lambda}\right) \Rightarrow \lambda = \frac{h}{m\,c}$$

If m is the mass of the particle and v is its velocity then

$$\lambda = \frac{h}{p} = \frac{h}{mv} \tag{2.5}$$

This wavelength is known as *de-Broglie wavelength*.

2.5 de-Broglie Wavelength for a Free Particle in Terms of its Kinetic Energy

The wavelength associated with a particle of mass m moving with velocity v can be obtained in similar manner as of photon of momentum p and energy E. For a free particle the total energy is same as its kinetic energy which is given by

$$E = \frac{1}{2}mv^2 = \frac{m^2v^2}{2\,m} \Rightarrow E = \frac{p^2}{2\,m} \quad [\because p = mv]$$

Now

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2\,mE}} \tag{2.6}$$

If a charged particle having charge q is accelerated through a potential difference of V volts then de-Broglie wave is

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mqv}} \quad [\because E = qv] \tag{2.7}$$

For an electron

$$\lambda = \frac{6.62\times10^{-34}\text{ J-sec}}{\sqrt{2\times9.0\times10^{-31}\text{ kg}\times1.6\times10^{-19}\text{ C}\times V}} = \frac{12.27}{\sqrt{V}}\,\text{Å} \tag{2.8}$$

According to kinetic theory of gases, the average kinetic energy of material particle is given by

$$E = \frac{1}{2}mv^2 = \frac{3}{2}\,k\,T$$

where k is Boltzmann's constant and T is absolute temperature of the particle. Now

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2m\,E}} = \frac{h}{\sqrt{3mkT}} \tag{2.9}$$

Thus, Eqs. (2.5)–(2.9) represent the de-Broglie wavelength of a wave associated with various moving particles.

2.6 Analysis of Matter Wave or de-Broglie Wave

de-Broglie's suggestion that with any moving material particle there is a wave associated with it brings the problem of reconciliation of the two seemingly different manifestation of energy, that is, particle having mass which is localized in space and time while waves being massless are de-localized in space and time. The two possible solutions are either to de-localize the particle (no existing theory suggest it) or to localize a wave which is quite plausible using Principle of Superposition which suggests that it is possible to create waves of almost any shape (wave packet) by adding sine waves with properly chosen wave numbers (k), amplitudes and phases.

Further, while deriving the expression for velocity of de-Broglie wave, we will show a contradiction and thus led towards the idea that material particle cannot be equivalent to a single wave.
Let m be the mass of the particle and v be its velocity. Then

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Also we have

$$E = mc^2 \text{ and } E = h\upsilon \quad \Rightarrow \upsilon = mc^2/h$$

Matter wave or de-Broglie wave velocity is

$$u = \nu\,\lambda = (mc^2/h)(h/mv) = c^2/v$$

Clearly for any material particle if $v < c$, then $u > c$ which is highly unexpected. Physically it means de-Broglie wave associated with a particle would travel faster than the particle itself, which indicates that material particle cannot be equivalent to a single wave.

Utilizing these above facts, one can visualize how a wave (not a single wave but a wave packet) resembles a particle. In other words, material particle (de-Broglie wave) in motion is equivalent to a wave packet rather than a single wave.

Wave packets have two velocities:

1. Phase velocity, $v_p = \omega/k$, by which individual wave constituting the packet moves.
2. Group velocity, $v_g = d\omega/dk$, by which the packet itself moves.

Clearly since phase velocity v_p (earlier u) is always greater than c, it is group velocity by which the particle moves.

To show $v_g = v$ we have angular frequency

$$\omega = 2\pi\nu = \frac{2\pi mc^2}{h}$$

and propagation constant

$$k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h}$$

of de-Broglie wave associated with a particle of rest mass m_0 and moving with velocity v. We also have

$$m = \frac{m_0}{\sqrt{1-(v^2/c^2)}}$$

Hence,

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{1-(v^2/c^2)}} \quad \text{and} \quad k = \frac{2\pi m_0 v}{h\sqrt{1-(v^2/c^2)}}$$

Now

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\left(1-\dfrac{v^2}{c^2}\right)^{3/2}} \quad \text{and} \quad \frac{dk}{dv} = \frac{2\pi m_0}{h\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}$$

The group velocity is given as

$$v_\mathrm{g} = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} = \frac{2\pi m_0 v \Big/ h\left(1-\dfrac{v^2}{c^2}\right)^{3/2}}{2\pi m_0 \Big/ h\left(1-\dfrac{v^2}{c^2}\right)^{3/2}} = v$$

Thus, a moving particle can be represented by a 'wave group' or 'wave packet'. Finally, the phase velocity v_p of the de-Broglie wave associated with a moving particle is given as

$$v_\mathrm{p} = \frac{\omega}{k} = \frac{E/\hbar}{p/\hbar} = \frac{E}{p} \tag{2.10}$$

The total relativistic energy 'E' of a particle is given by

$$E = \sqrt{p^2c^2 + m_0^2c^4} \tag{2.11}$$

Substituting this value in Eq. (2.10), we have

$$v_\mathrm{p} = \frac{\sqrt{p^2c^2 + m_0^2c^4}}{p} = \frac{pc\sqrt{1+\dfrac{m_0^2c^2}{p^2}}}{p}$$

But $p = h/\lambda$. Therefore,

$$v_\mathrm{p} = c\sqrt{1+\left(\frac{m_0 c\lambda}{h}\right)^2} \tag{2.12}$$

This equation shows that the phase velocity of a wave associated with a moving particle is always greater than c and even in free space it is a function of λ.

2.7 Davisson and Germer Experiment

In order to confirm the de-Broglie hypothesis, Davisson and Germer performed an experiment in 1927 to observe the diffraction of electrons. From this experiment the wavelength of the diffracted electrons calculated from the diffraction pattern is nearly same as the value calculated from the de-Broglie hypothesis. This confirmed the de-Broglie hypothesis of matter waves.

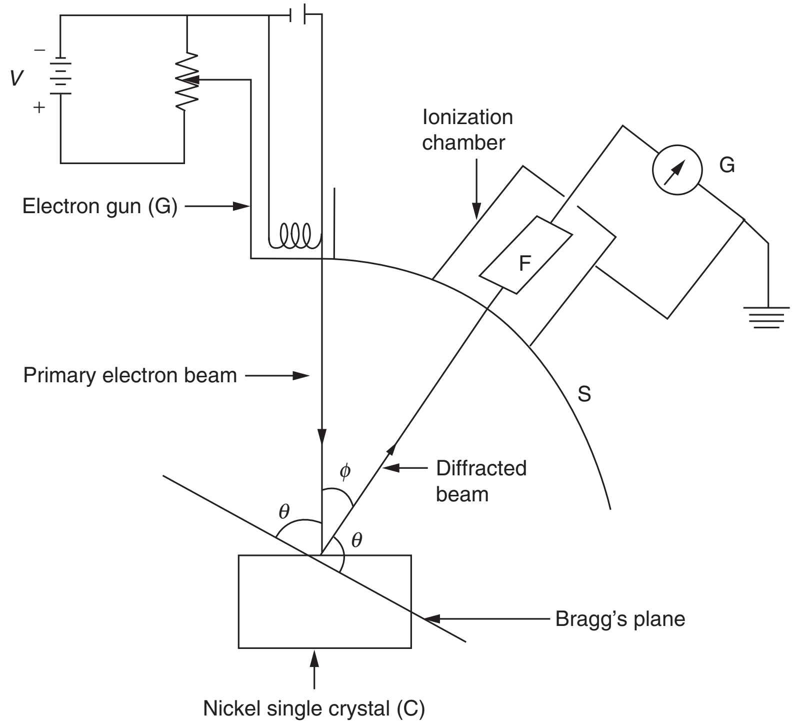


Figure 1 Davisson and Germer experiment.

2.7.1 Experimental Setup

The Davisson and Germer apparatus is shown in Fig. 1. Electrons from a heated filament are accelerated through a variable potential 'V' and emerge from the electron gun 'G' (variable electron energy). The electron beam falls normally on a nickel crystal 'C' (variable angle of incidence). The electrons are diffracted (read scattered) from the crystal in all directions. The intensity of the diffracted beam (read collection of electrons) in different directions is measured by a Faraday cylinder 'F' which is connected to a galvanometer and can move on a circular scale 'S' (variable detector position). The crystal can be turned about an axis parallel to the incident beam. The whole apparatus is enclosed in an evacuated chamber.

2.7.2 Method

The accelerating potential V is given a small value and the crystal is turned at any arbitrary azimuth. The Faraday cylinder is moved to various positions on a scale 'S' and the galvanometer current is noted at each position (see Fig. 1). The current is a measure of intensity of diffracted beam (read amount of electron collected). The current is then plotted against the angle ϕ (angle between incident beam and the bean entering the cylinder). The observations are repeated for different accelerating potential and the corresponding curves are plotted. One such curve for $V = 54$ V between current and angle ϕ is shown in Fig. 2 (see and verify the similarity between this and the figure in Bragg's spectrometer (crystal-structure).

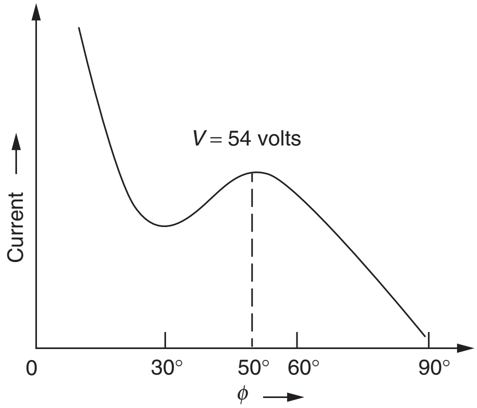


Figure 2 The intensity of the diffracted beam plotted against the diffracting angle ϕ for $V = 54$ V.

2.7.3 Observations and Calculations

It has been observed that intensity of scattered beam is maximum at $\phi = 50°$ with the original beam but incident and scattered beam makes an angle of $\theta = 65°$ (180° − 50° = 130°, 130°/2 = 65°) with Bragg's plane when the accelerating voltage $V = 54$ V. Therefore, the theoretical value of λ is given by de-Broglie hypothesis

$$\lambda = \frac{12.28}{\sqrt{V}}\,\text{Å} \Rightarrow \lambda = \frac{12.28}{\sqrt{54}}\,\text{Å} = 1.67\ \text{Å} \tag{2.13}$$

The experimental value of λ as calculated using Bragg's law

$$2d \sin\theta = n\lambda \tag{2.14}$$

where θ is angle of diffraction (as mentioned and measured in Bragg's Law) and d is the interplanar spacing which is 0.91 Å in this case (measured independently by x-ray diffraction).
For 1st order, $n = 1$,

$$\lambda = 2 \times 0.91\ \text{Å} \times \sin 65° = 1.65\ \text{Å} \tag{2.15}$$

which is very nearly equal to the value of λ calculated from de-Broglie relation. This shows that electrons behave like wave in certain circumstances.

Davisson–Germer experiment, thus, provides direct verification of de-Broglie hypothesis.

2.8 Bohr's Quantization Condition

Another remarkable feature of de-Broglie hypothesis is that it beautifully fits in the Bohr's atomic model and the famous Bohr's quantization can easily be established. The wave character of electron can account for the limited number of permissible orbits in an atom and also for the quantized angular momentum $\vec{L}$ of an electron in an atom.

According to Bohr's theory of Hydrogen atom, the electron revolves in non-radiating circular orbits around the nucleus. Since the electron does not radiate energy while moving in its orbit, the wave associated with it must be a stationary wave in which there is no transference of energy (Fig. 3).

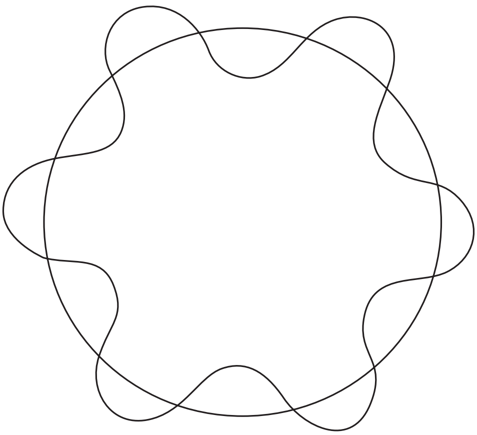


Figure 3 Interpretation of Bohr's quantization rule on the basis of de-Broglie wave.

Using Bohr's theory of Hydrogen atom, the velocity of electron in the 1st Bohr's orbit is given by

$$v = \frac{e}{\sqrt{4\pi\varepsilon_0 mr}}$$

that is

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = \frac{mv^2}{r}$$

Now the de-Broglie wavelength associated with this electron is

$$\lambda = \frac{h}{mv} = \frac{h}{e}\sqrt{\frac{4\pi\varepsilon_0 r}{m}}$$

Substituting the values of h, e, m, ε_0 and $r = 5.3 \times 10^{-11}$ m (radius of 1st Bohr's orbit) we get $\lambda = 33 \times 10^{-11}$m. Now the circumference of 1st Bohr's orbit is

$$2\pi r = 2 \times 3.14 \times 5.3 \times 10^{-11} = 33 \times 10^{-11}\text{m}$$

which is same as the wavelength of de-Broglie wavelength associated with the electron moving in 1st Bohr's orbit. Hence, it can be concluded that the electron wave can be adjusted around an orbit only when the circumference of its orbit is an integral multiple of the wavelength. This means that only those orbits are permitted for which

$$2\pi r = n\lambda = \frac{nh}{mv} \quad \Rightarrow mvr = \frac{nh}{2\pi} \tag{2.16}$$

But angular momentum is given by

$$L = mvr \tag{2.17}$$

From Eqs. (2.16) and (2.17)

$$L = \frac{nh}{2\pi} \tag{2.18}$$

which is the famous *Bohr's quantization condition*.

2.9 Phase Velocity and Group Velocity

When plane waves of different wavelengths travel simultaneously in the same direction along a straight line through a dispersive medium (a medium in which the phase velocity of a wave depends on its wavelength), successive groups of the waves are produced. These wave groups are called 'wave packet'. *The velocity of each individual wave of a wave packet is known as phase velocity*. The phase velocity is also called 'wave velocity' or 'velocity of propagation'. It is denoted by v_p and defined as

$$v_p = \frac{\omega}{k}$$

where ω is the angular velocity and k is the propagation constant of the wave.

The average velocity through which the wave packet propagates in the medium is called group velocity (v_g). The group velocity may also be defined as the velocity with which the energy in the group is transmitted. However, the individual waves travel inside the group with their own velocities. Mathematically, group velocity is given by

$$v_g = \frac{d\omega}{dk}$$

2.9.1 Expression for Phase Velocity

A plane harmonic wave travelling along the positive x-direction is given by

$$y = a\sin(\omega t - kx) \tag{2.19}$$

where a is the amplitude, $\omega = 2\pi n$ is the angular frequency and $k = 2\pi/\lambda$ is the propagation constant.

By definition, the ratio of angular frequency ω to propagation constant k is wave velocity. Thus

$$v_p = \frac{\omega}{k} \tag{2.20}$$

In Eq. (2.19), $(\omega t - kx)$ is the phase of wave motion. Then the planes of constant phase (wave front) are defined as

$$\omega t - kx = \text{constant} \tag{2.21}$$

Differentiating w.r.t. t, we get

$$\omega - k\frac{dx}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$$

but

$$\frac{dx}{dt} = v_p \quad \Rightarrow v_p = \frac{\omega}{k} \tag{2.22}$$

Equation (2.22) is the required expression for phase velocity.

2.9.2 Expression for Group Velocity

Let us consider a wave group which consists of two components of equal amplitude 'a' and slightly different angular frequencies ω_1 and ω_2 and propagation constants k_1 and k_2, respectively. Their separate displacement are given by

$$y_1 = a\sin(\omega_1 t - k_1 x) \tag{2.23}$$

$$y_2 = a\sin(\omega_2 t - k_2 x) \tag{2.24}$$

Using principle of superposition we have

$$\begin{aligned} y &= y_1 + y_2 = a\,[\sin(\omega_1 t - k_1 x) + \sin(\omega_2 t - k_2 x)] \\ &= 2a \sin\left(\frac{\omega_1 t - k_1 x + \omega_2 t - k_2 x}{2}\right)\cos\left(\frac{\omega_1 t - k_1 x - \omega_2 t + k_2 x}{2}\right) \\ &= 2a \sin\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right]\cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \\ &= 2a \cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right]\sin\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right] \end{aligned} \tag{2.25}$$

Equation (2.25) represents a wave group of frequency $(\omega_1 + \omega_2)/2$ and amplitude

$$A = 2a \cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \tag{2.26}$$

Thus, the amplitude of the wave group is modulated both in space and time by a very slowly varying envelope of frequency $(\omega_1 - \omega_2)/2$ and propagation constant $(k_1 - k_2)/2$. The maximum value of amplitude is $2a$. This envelope is represented by the dotted curve as shown in Fig. 4.

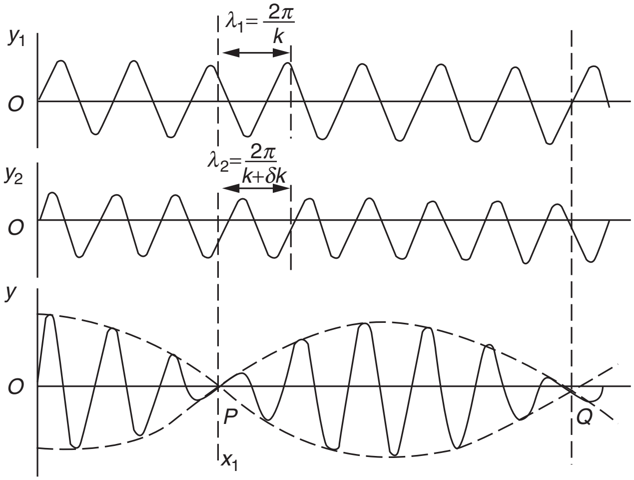


Figure 4 Modulation of wave.

The velocity with which this envelope advances, that is, the velocity of maximum amplitude of the group is given by

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k} \tag{2.27}$$

For infinitesimally small frequency interval, we can write

$$v_g = \frac{d\omega}{dk} \tag{2.28}$$

Equation (2.28) is the expression for group velocity.

2.9.3 Relation between Phase Velocity (v_p) and Group Velocity (v_g)

The group velocity is given by

$$v_{\mathrm{g}} = \frac{d\omega}{dk} \tag{2.29}$$

and the phase velocity is given by

$$v_{\mathrm{p}} = \frac{\omega}{k} \tag{2.30}$$

or

$$\omega = kv_{\mathrm{p}} \tag{2.31}$$

Substituting the value of ω in Eq. (2.29), we have

$$v_{\mathrm{g}} = \frac{d(kv_{\mathrm{p}})}{dk} = k\frac{dv_{\mathrm{p}}}{dk} + v_{\mathrm{p}}$$

But $k = 2\pi/\lambda$. This implies

$$dk = -\frac{2\pi}{\lambda^2}d\lambda$$

Therefore,

$$v_{\mathrm{g}} = v_{\mathrm{p}} - \frac{2\pi}{\lambda}\frac{\lambda^2}{2\pi}\frac{dv_{\mathrm{p}}}{d\lambda} \quad \Rightarrow v_{\mathrm{g}} = v_{\mathrm{p}} - \lambda\frac{dv_{\mathrm{p}}}{d\lambda} \tag{2.32}$$

Equation (2.32) is the required expression. Equation (2.32) shows that $v_{\mathrm{g}} < v_{\mathrm{p}}$ when the medium is dispersive, that is, when v_{p} is the function of λ. If there is no dispersive medium, that is, waves of all the wavelengths travel with same speed then $dv_{\mathrm{p}}/d\lambda = 0$. So, Eq. (2.32) gives $v_{\mathrm{g}} = v_{\mathrm{p}}$. This result holds for elastic waves in homogeneous medium and electromagnetic waves in vacuum.

2.10 Phase Velocity of de-Broglie Waves

Let ω be the angular frequency and k be the propagation constant of de-Broglie wave. Then the phase velocity v_{p} of this wave is given by

$$v_{\mathrm{p}} = \frac{\omega}{k} \tag{2.33}$$

According to de-Broglie hypothesis, the energy E and momentum p of a particle is given by

$$E = \hbar\omega \quad \text{and} \quad p = \hbar k$$

So,

$$E/p = \omega/k \quad \text{or} \quad v_{\mathrm{p}} = E/p \tag{2.34}$$

Let m be the relativistic mass of the particle and v be its velocity. Then

$$E = mc^2 \quad \text{and} \quad p = mv$$

So,

$$v_{\mathrm{p}} = mc^2/mv \quad \text{or} \quad v_{\mathrm{p}} = c^2/v \tag{2.35}$$

But the particle velocity v is always less than c. Therefore, the phase velocity v_{p} is always greater than c. This is an unexpected result. According to this result, the wave associated with the moving particle would travel much faster than the particle itself and would leave the particle far behind. This is nothing but the failure of wave description of the particle. Therefore, the phase velocity has no physical meaning and the particle in motion is always associated with a wave packet.

2.11 Heisenberg's Uncertainty Principle

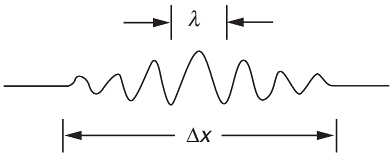
Heisenberg in 1927 proposed an uncertainty principle which was direct consequence of dual nature of matter. It is well known that in classical mechanics a particle is described by its definite momentum (velocity) and position. Therefore, its position and velocity (momentum) both can be determined with desired accuracy. However, in quantum mechanics a particle is described by a wave packet moving with group velocity and particle can be located anywhere within the limit of wave packet. Therefore, position of particle in the packet is not certain. Further, the wave packet has a velocity spread; hence, the momentum of particle is also uncertain. For example, if the wave packet is large, the velocity spread is very small and it can be determined accurately, but at the same time the position in such large packet is very uncertain and cannot be determined accurately. On the other hand, when wave packet is small, the position of the particle is almost fixed, but the velocity spread is very large, hence very uncertain. Therefore, certainty in position and in velocity simultaneously is not possible in quantum mechanics. It can be demonstrated in the following way.

While analyzing the matter wave earlier we only developed the idea about how to bring two seemingly different aspects such as particle and wave at convergence and for that the only plausible explanation was – localizing a wave in space and time. Utilizing Principle of Superposition, wave packet having constructive interference over a small region and destructive elsewhere (requires a great number of waves with slightly different amplitude, wave number and phases) resembles a particle.

Here, it is also worth mentioning that space localization can be demonstrated as visualized space axes (x, y, z), but time localization cannot be demonstrated as time axis (t) and is not so easy to visualize. Hence, we take the help of analogies.

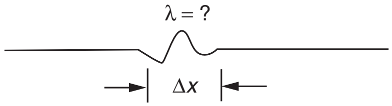
Now, a wave in space can be of two types:

Type (a): Diffuse wave packet is possible only when large number of waves with single wave number (k) superimpose.



$\Delta k \to 0$ and $\Delta x \to \infty$

Type (b): Sharp wave packet is possible only when large number of waves with different wave number (k) superimpose.



$\Delta k \to \infty$ and $\Delta x \to 0$

From the above two types, it is obvious that Δk and Δx are inversely related. If $\Delta k \to 0$, then $\Delta x \to \infty$ or if $\Delta k \to \infty$, then $\Delta x \to 0$. In other words, single k (certain k) gives diffuse wave packet (uncertain position) and indefinite k (uncertain k) generates sharp wave packet (certain position).

Thus, finally localizing the wave in space we get

$$\Delta k \, \Delta x \approx 1 \tag{2.36}$$

Likewise, localizing the wave in time we get

$$\Delta\omega\,\Delta t \approx 1 \tag{2.37}$$

As discussed earlier, space variables are replaced by corresponding time variables while deriving expression for localized wave in time. In other words, $k = 2\pi/\lambda$ (space) is replaced by $\omega = 2\pi/T$ (time) (also see, T is time corresponding to λ space) and x by t. Thus, the requirement of localizing a wave in space and time further extends this idea, wherein uncertainty creeps in as one goes down to the realm of particles having size of the order of de-Broglie wavelength as Heisenberg's Uncertainty Principle for position and momentum ($\Delta x\ \Delta p \geq \hbar$) with space localization of wave and energy and time ($\Delta E\ \Delta t \geq \hbar$) with time localization of wave, respectively.

Now, for uncertainty principle of position and momentum we have

$$\Delta\ k = \Delta\ (2\pi/\lambda) = \Delta\ (2\pi p/h) = (2\pi/h)\ \Delta p \qquad \text{using de-Broglie expression}$$

Finally, using Eq. (2.36) we get

$$(2\pi/h)\Delta p\ \Delta x \approx 1$$

or

$$\Delta p\ \Delta x \approx (h/2\pi) = \hbar \qquad \text{(reduced Planck's constant)}$$

The above condition assumes ideal instruments (with unlimited accuracy – not practical) but in practice it is still worse, hence, '$\approx$' should be replaced by '$\geq$', that is, $\Delta x\ \Delta p \geq \hbar$.

Therefore, Heisenberg's uncertainty principle can be defined as it is impossible to determine the exact position and momentum of a particle simultaneously.

Likewise for uncertainty principle for energy and time, we have

$$\Delta\omega = \Delta(2\pi/T) = \Delta(2\pi v) = 2\pi\ \Delta v = 2\pi(\Delta E/h) \qquad \text{using Planck's Expression}$$

Finally, using Eq. (2.37) we get

$$2\pi(\Delta E/h)\ \Delta t \approx 1$$

or

$$\Delta E\ \Delta t \approx h/2\pi = \hbar$$

Just as above $\Delta E\ \Delta t \geq \hbar$. *It is not possible to determine both the energy and time co-ordinate of a particle with unlimited precision.*

2.11.1 Physical Significance of Heisenberg's Uncertainty Principle

The uncertainty principle leads to the following conclusions:

1. If we measure the position 'x' of a particle then uncertainty in position Δx vanishes while uncertainty in momentum becomes infinite and vice-versa. Thus, if we design an experiment to measure 'x' or 'p_x' accurately, the other quantity will become completely uncertain.
2. We can measure both the quantities by means of experiment but only to a certain limit of accuracy specified by the uncertainty principle.
3. For a particle of mass 'm' moving with velocity 'v', the product of uncertainty Δx and uncertainty Δv is given by $\Delta x\ \Delta v \approx \hbar/m$. For a heavy particle $\hbar/m$ is very small and therefore, the uncertainty product $\Delta x\ \Delta v$ becomes very small. For such a particle both the position 'x' and velocity 'v' can be determined accurately. For heavy bodies, if 'm' is such that $\hbar/m \approx 0$, then the uncertainty vanishes and all quantities can be determined with perfect accuracy. This is the limiting case of classical mechanics.

Thus classical mechanics is true for heavy bodies and the uncertainties are characteristics of quantum mechanics which is applicable to light particles such as electron, proton, neutron etc.

2.11.2 Examples of Position-Momentum Uncertainty

2.11.2.1 Determination of the Position of a Particle by Heisenberg's Power Gamma Ray Microscope

In order to determine the position of an electron, let us take a high resolving power microscope. If Δx is the smallest distance between the two points which can be resolved through microscope, then

$$\Delta x = \frac{\lambda}{2\sin\theta} \tag{2.38}$$

where λ is the wavelength of light used and θ is the semi-vertical angle of the cone as shown in Fig. 5.

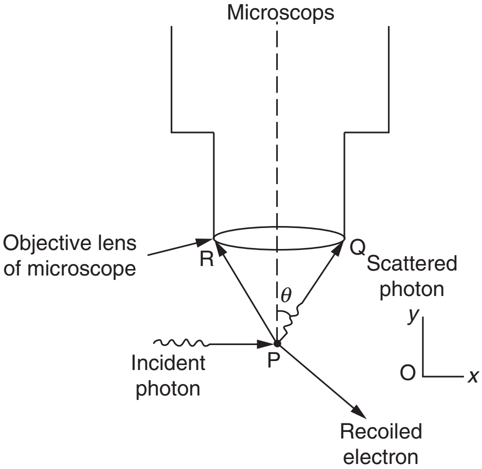


Figure 5 Heisenberg's gamma ray microscope.

The electron is seen by the γ-radiation scattered by it into the microscope. As a result, there will be a change in its momentum due to recoiling of the electron. Consider a scattered photon of wavelength λ which enters the objective of the microscope anywhere between PQ and PR. Then the momentum of the scattered photon will be h/λ. If the scattered photon enters the objective in the direction of PR, then its component in the direction of x-axis will be $(h/\lambda)\sin\theta$.

If λ' is the wavelength of γ-radiation before scattering, then the momentum along x-axis is

$$p_x = \frac{h}{\lambda'} - \frac{h}{\lambda}\sin\theta$$

Now, when the scattered photon enters the objective in the direction of PQ, its component in the direction of x-axis will be $-(h/\lambda)\sin\theta$. So the momentum of the electron along x-axis is

$$p_x = \frac{h}{\lambda'} + \frac{h}{\lambda}\sin\theta$$

Therefore, the uncertainty in measurement of the momentum of the electron is given by

$$\Delta p_x = \left(\frac{h}{\lambda'} + \frac{h}{\lambda}\sin\theta\right) - \left(\frac{h}{\lambda'} - \frac{h}{\lambda}\sin\right) = \frac{2h}{\lambda}\sin\theta \tag{2.39}$$

From Eqs. (2.38) and (2.39), we have

$$\Delta x \Delta p_x = \frac{\lambda}{2\sin\theta} \times \frac{2h}{\lambda}\sin\theta = h$$

Therefore

$$\Delta x \ \Delta p \geq \hbar$$

This is in accordance with the Heisenberg's uncertainty principle.

2.11.2.2 Diffraction of Electron Beam by a Single Slit

Let us consider a parallel beam of electron falling normally on a narrow slit AB of width Δx. Diffraction will occur at the slit and a central bright band surrounded by alternate bright and dark bands of decreasing intensity on both of its sides can be obtained on photographic plate which is situated at some distance from the slit as shown in Fig. 6. The first minimum of the diffraction ($2d\sin\theta = n\lambda$) is given by

$$\Delta x \sin\theta = \lambda$$

where $2d = \Delta x$, $n = 1$ and θ is the angle through which the electron beam is diffracted by the slit.

Since determination of the exact position of the slit where the electron beams enters is uncertain, therefore, the uncertainty in measurement of the position of the electron along y-axis is given by

$$\Delta x = \frac{\lambda}{\sin\theta} \tag{2.40}$$

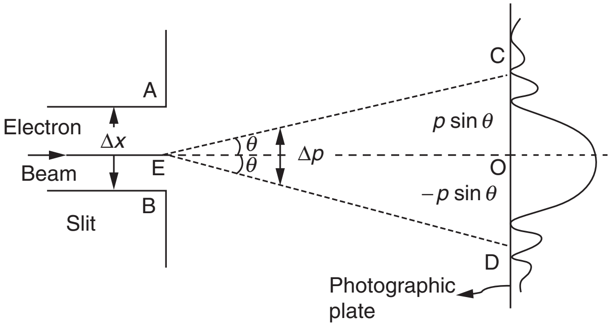


Figure 6 Diffraction of electron by single slit.

Let p be the momentum of the electron. After diffraction, the electron moves along EC and ED. The momentum of the diffracted electron along OC and OD will be $p\sin\theta$ and $-p\sin\theta$, respectively. As momentum can lie anywhere in between $p\sin\theta$ and $-p\sin\theta$, therefore the uncertainty in measurement of the momentum of the electron along y-axis is given by

$$\Delta p = (p\sin\theta) - (-p\sin\theta) = 2p\sin\theta$$

or

$$\Delta p = 2\frac{h}{\lambda}\sin\theta \qquad \left(\because p = \frac{h}{\lambda}\right) \tag{2.41}$$

From Eqs. (2.40) and (2.41), we have

$$\Delta x \Delta p_x = \frac{\lambda}{\sin\theta} \times \frac{2h}{\lambda} \sin\theta = 2h$$

or

$$\Delta x \Delta p \geq \hbar$$

2.11.3 Applications of Uncertainty Principle

2.11.3.1 Non-Existence of Free Electron in Nucleus

Let us suppose that the electron is present inside the nucleus whose radius is typically of the order of 0.5×10^{-14} m. Hence, its diameter would be 10^{-14} m. Therefore, the maximum uncertainty in the position of electron is nothing but the diameter of the nucleus, that is, $\Delta x = 10^{-14}$ m.

Then, according to uncertainty principle, the uncertainty in momentum is given by

$$\Delta p_{\min} \geq \frac{\hbar}{\Delta x_{\max}}$$

Therefore, minimum uncertainty in momentum

$$\Delta p_{\min} = \frac{\hbar}{\Delta x_{\max}} = \frac{6.62 \times 10^{-34}}{6.28 \times 10^{-14}} = 1.05 \times 10^{-20} \text{ kg m/sec}$$

Further, it is also important to note that minimum uncertainty in momentum physically means momentum itself, that is, $\Delta p_{\min} = p$. Now, the total relativistic energy of a particle is given by

$$E = \sqrt{p^2c^2 + m_0^2c^4}$$

As the rest mass energy m_0c^2 of an electron is only 0.51 MeV, which is very small and hence can be neglected, therefore, we have

$$E = pc \text{ or } E_{\min} = P_{\min}\, c = 1.05 \times 10^{-20} \times 3 \times 10^{8} = 3.15 \times 10^{-12} \text{ Joules} = 19.6 \text{ MeV}$$

Thus, if a free electron exists inside the nucleus, it must have a minimum energy of about 19.6 MeV but experiments show that the β-particle (i.e. electron) emitted from a radioactive nuclei can have a maximum energy of about 4 MeV. It shows that the theoretical and experimental results do not agree at all. Therefore, it can be concluded that ***a free electron cannot exist inside the nucleus***.

2.11.3.2 Existence of Protons, Neutrons within the Nucleus

Let us suppose that a proton, neutron or α-particle is present inside the nucleus. Then the maximum uncertainty in the position of proton, neutron and α-particle is nothing but the diameter of the nucleus, that is, $\Delta x = 2 \times 10^{-14}$ m. Then, according to uncertainty principle, the uncertainty in momentum is given by

$$\Delta p_{\min} \geq \frac{\hbar}{\Delta x_{\max}}$$

Therefore, minimum uncertainty in momentum

$$\Delta p_{\min} = \frac{\hbar}{\Delta x_{\max}} = \frac{6.62 \times 10^{-34}}{2\pi \times 2 \times 10^{-14}} = 5.3 \times 10^{-21} \text{ kg m / sec}$$

The corresponding energies of the proton, neutron or α-particle are given by

$$E = \frac{p^2}{2m} = \frac{(5.3 \times 10^{-21})^2}{2 \times 1.6 \times 10^{-27}} = 8.4 \times 10^{-15} \text{ J} = \frac{8.4 \times 10^{-15}}{1.6 \times 10^{-19}} = 52 \text{ keV}$$

Since, the energies carried by the protons or neutrons emitted by the nuclei are higher than 52 keV and for α-particles more than 13 keV, so these particles can exist in the nuclei.

2.12 Schrödinger Wave Equation

Schrödinger, in 1926, for the first time, gave the mathematical formulation to deal with the wave situation existing at the microscopic level, that is, the mathematical description of matter waves. The significance of Schrödinger equation is that it plays the same role in wave mechanics as the Newton's laws for classical mechanics. Schrödinger equation was formulated as the differential equation for the de-Broglie waves associated with the particles and describes the motion of particles.

Schrödinger has developed two wave equations known as time-independent and time-dependent Schrödinger wave equations. The applicability of wave equation depends upon the nature of physical problems. Here, we focus on the application of time-independent wave equation such as particle in one-dimensional box.

2.12.1 Time-Independent Schrödinger Wave Equation

The wave equation is a second-order differential equation whose solution gives the wave disturbances in the medium. The differential equation that can describe any kind of wave motion is given as

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{2.42}$$

Here u is any function and v is the velocity of wave. This equation is capable enough to describe the motion of all types of wave, for example, mechanical waves, EM waves and matter waves.

To describe the matter waves by using Eq. (2.42), Schrödinger introduced a mathematical function ψ which is the variable quantity associated with the moving particle. It is a complex function of space co-ordinate and time of the particle. ψ is known as **wave function** which describes the de-Broglie waves associated with the moving particle. So, for de-Broglie waves, Eq. (2.42) can be written as

$$\nabla^2 \psi\ (r, t) = \frac{1}{v^2}\frac{\partial^2 \psi(r, t)}{\partial t^2} \tag{2.43}$$

The solution of Eq. (2.43) is given as

$$\psi(r, t) \ = \ \psi_0(r)\ e^{-i\omega t} \tag{2.44}$$

Partially differentiating Eq. (2.44) twice w.r.t. 't' we get

$$\frac{\partial \psi(r, t)}{\partial t} = -i\omega\psi_0(r)e^{-i\omega t} \quad \Rightarrow \quad \frac{\partial^2 \psi(r, t)}{\partial t^2} = -\omega^2\psi_0(r)e^{-i\omega t} = -\omega^2\psi(r,t) \tag{2.45}$$

From Eqs. (2.43) and (2.45) we get

$$\nabla^2 \psi\ (r, t) = -\frac{\omega^2}{v^2}\ \psi(r,t) = -\frac{4\pi^2}{\lambda^2}\ \psi(r,t) \quad \Rightarrow \quad \nabla^2 \psi\ (r,\ t) + \frac{4\pi^2}{\lambda^2}\ \psi(r,t) \ = \ 0 \tag{2.46}$$

So far we have not used the concept of matter waves. Now to include the concept of matter waves, we put $\lambda = h/mv$ in Eq. (2.46) and get

$$\nabla^2\psi\ (r,t) + \frac{4\pi^2 m^2 v^2}{h^2}\psi(r,t) = 0 \tag{2.47}$$

Let E and V be the total energy and potential energy, respectively. Then the kinetic energy is given by

$$K = \frac{1}{2}mv^2 = E - V = m^2v^2 = 2m\,(E - V)$$

Putting value of m^2v^2 in Eq. (2.47) we get

$$\nabla^2\psi(r,t) + \frac{8\pi^2 m(E-V)}{h^2}\psi(r,t) = 0$$

or

$$\nabla^2\psi(r,t) + \frac{2m(E-V)}{\hbar^2}\ \psi(r,t) = 0 \tag{2.48}$$

Here $\hbar = h/2\pi$ is known as '*reduced Planck's constant*'. This is time-independent Schrödinger wave equation.

2.12.2 Time-Dependent Schrödinger Wave Equation

For time-dependent Schrödinger wave equation, partially differentiating Eq. (2.44) w.r.t. '*t*' we get

$$\frac{\partial\psi(r,t)}{\partial t} = -i\omega\psi_0(r)e^{-i\omega t} = -\ i(2\pi\nu)\psi_0(r)e^{-i\omega t} = -2\pi i\frac{E}{h}\psi(r,t) = -i\frac{E}{\hbar}\psi(r,t)\frac{i}{i}$$

$$\Rightarrow E\psi(r,t) = i\ \hbar\frac{\partial\psi(r,t)}{\partial t} \tag{2.49}$$

Substituting value of $E\psi(r,t)$ in time-independent Schrödinger wave equation [Eq. (2.48)], we get

$$\nabla^2\psi(r,t) + \frac{2\ m}{\hbar^2}\left[i\hbar\frac{\partial\psi(r,t)}{\partial t} - V\psi(r,t)\right] = 0$$

$$\Rightarrow \nabla^2\psi(r,t) = -\frac{2\ m}{\hbar^2}\left[i\hbar\frac{\partial\psi(r,t)}{\partial t} - V\psi(r,t)\right] = 0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi(r,t) + V\ \ \psi(r,t) = i\hbar\frac{\partial\psi(r,t)}{\partial t}$$

$$\Rightarrow \left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi(r,\ t) = i\hbar\frac{\partial\psi(r,t)}{\partial t} \tag{2.50}$$

Equation (2.50) is known as 'time-dependent Schrödinger wave equation'. This equation is used in explaining the non-stationary phenomenon such as electronic transition between two states of an atom. Here $[-(\hbar^2/2m)\nabla^2 + V]$ is known as 'Hamiltonian' and is denoted by 'H'. It represents the total energy of the system. $i\hbar(\partial/\partial t)$ is energy operator which when operated on $\psi(r,t)$ gives the energy.

2.12.3 Derivation of Time-Independent Equation from Time-Dependent Equation

The time-dependent Schrödinger wave equation is as follows:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi(r,t) = i\hbar\frac{\partial\psi(r,t)}{\partial t}$$

For one-dimensional Schrödinger wave equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Since $i\hbar(\partial/\partial t)$ is energy operator, therefore we can replace it by E. So

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V\psi = E\psi \quad \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi - V\psi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = -\frac{2m}{\hbar^2}(E-V)\,\psi \quad \Rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E-V)\,\psi = 0$$

This is Schrödinger one-dimensional time-independent wave equation.

2.13 Physical Interpretation of Wave Function ψ

According to Max Born, the square of the magnitude of the wave function, that is, $|\psi|^2$ (or $\psi\psi^*$ if ψ is a complex quantity) calculated at a particular point represents the probability of finding the particle at that particular point. $|\psi|^2$ is also known as probability density while ψ is called probability amplitude.

According to this interpretation, the probability of finding a particle within an arbitrary volume element $d\tau$ is $|\psi|^2\, d\tau$. Since the particle will definitely present somewhere within this volume element, therefore the integral of $|\psi|^2\, d\tau$ over the entire space must be unity, that is, $\int_{-\infty}^{+\infty} |\psi|^2\, d\tau = 1.$ A wave function which satisfies this equation is known as **normalized wave function**.

2.14 Normalized Wave Function

Let us consider the case in which the motion of the particle is bounded to a limited region by some external forces. In such cases, the probability of finding the particle over the entire region will be unity. Since probability of finding a particle over a small volume element $d\tau$ is $|\psi(\vec{r},t)|^2\, d\tau$, therefore, probability of finding the particle over the entire space will be

$$\int |\psi(\vec{r},t)|^2 d\tau = 1 \quad \text{or} \quad \int \psi(\vec{r},t)\ \psi^*(\vec{r},t)\ d\tau = 1 \tag{2.51}$$

Those wave functions that satisfy Eq. (2.51) are known as normalized wave functions. If a wave function is not normalized then it can easily be normalized in the following manner:

Let $\int \psi(\vec{r},t)\ \psi^*(\vec{r},t)\ d\tau = N$. Then

$$\frac{1}{N}\int \psi(\vec{r},t)\ \psi^*(\vec{r},t)\ d\tau = 1 \Rightarrow \int \frac{\psi}{\sqrt{N}}\frac{\psi^*}{\sqrt{N}} d\tau = 1 \quad \Rightarrow \int \psi_1\ \psi_1^*\ d\tau = 1$$

where $\psi/\sqrt{N} = \psi_1$ and $\psi^*/\sqrt{N} = \psi_1^*$. Here $\sqrt{N}$ is called normalization factor. Since $1/\sqrt{N}$ is a constant and does not depend upon space co-ordinate (x, y, z) and ψ is a solution of wave equation, therefore, $\psi/\sqrt{N}$ will also be a solution of wave equation and will describe the same physical system.

2.15 Properties of Wave Function

Apart from being normalizable, a well-behaved or acceptable wave function must fulfill the following conditions:

1. ***The wave function ψ must be finite everywhere*** because if ψ is infinite at a particular point then it means that there is an infinitely larger probability of finding the particle at that particular point which is clear violation of uncertainty principle.
2. ***The wave function ψ must be single valued*** because if ψ has more than one values at a particular point then it means there is more than one value of probability of finding the particle at that particular point, which is not possible.
3. ***The wave function ψ must be continuous and must have continuous first-order derivative everywhere***. This property is obvious from Schrödinger equation which itself shows that $d^2\psi/dx^2$ must be finite everywhere. This is possible only if $d\psi/dx$ has no discontinuity at any boundary where potential can change. Finally, if $d\psi/dx$ is continuous then ψ will obviously be continuous.

2.16 Eigenvalues and Eigenfunctions

When the Schrödinger time-independent wave equation is solved for any particle using its wave function under certain boundary conditions, the solutions exist only for particular values of energy E_n which are known as the eigenvalues and the corresponding wave function ψ_n to the eigenvalues is called eigenfunction.

2.17 Applications of Schrödinger Wave Equations

2.17.1 Free Particle

A particle is said to be free in certain region if its potential energy 'V' is zero ($V = 0$), that is no net force is acting on the particle in that region. For such a region, time-independent Schrödinger wave equation can be written as

$$\nabla^2\psi\ (r,t) + \frac{2m(E-V)}{\hbar^2}\ \psi(r,t) = 0$$

which reduces into the following Schrödinger wave equation for free particle:

$$\nabla^2\psi(r,t) + \frac{2mE}{\hbar^2}\psi(r,t) = 0$$

Let the particle move in positive x-direction [here $\psi(r,t)$ is replaced by $\psi(x,t)$]. Without any loss of generality, as the particle is restricted to move only along x-axis keeping y- and z-axis constant, ∇^2 can be reduced to $\partial^2/\partial x^2$ which in this situation is nothing but d^2/dx^2. Then the wave associated with it can be represented by

$$\frac{d^2\psi(x,t)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x,t) = 0 \quad \Rightarrow \quad \frac{d^2\psi(x,t)}{dx^2} + k^2\psi(x,t) = 0 \tag{2.52}$$

where $k^2 = 2mE/\hbar^2$. The general solution of Eq. (2.52) is

$$\psi(x,t) = Ae^{ikx} + Be^{-ikx} \tag{2.53}$$

Here the term Ae^{ikx} represents the wave travelling in the positive x-direction whereas the term Be^{-ikx} represents the wave travelling in the negative x-direction, that is, reflected wave. As the particle is free and there is no boundary by which reflection can take place, so the term Be^{-ikx} has no significance. So Eq. (2.53) reduces to

$$\psi(x,t) = Ae^{ikx} \tag{2.54}$$

Equation (2.54) is used to describe the wave associated with the free particle moving in positive x-direction. In three dimensions, Eq. (2.54) can be written as

$$\psi(r,t) = Ae^{i(k_x x + k_y y + k_z z)} = Ae^{i\vec{k}\cdot\vec{r}} \tag{2.55}$$

2.17.2 Particle in One-Dimensional Infinitely Deep Potential Well (Or Particle in 1D Box)

If one-dimensional (1D) motion of particle is assumed to take place in a given region such that its potential energy is zero within this region and infinity at the extremities and outside of this region, then it is described as **'Particle in 1D Box'**.

Let a particle of mass m travel along x-axis bouncing back and forth between the walls of the box (see Fig. 7). The box is supposed to have walls of infinite height at $x = 0$ and $x = L$. In terms of boundary conditions imposed by the problem, the potential function is given as

$$V(x) = \begin{cases} \infty & \text{for} \quad x \geq L \text{ and } x \leq 0 \\ 0 & \text{for} \quad 0 < x < L \end{cases}$$

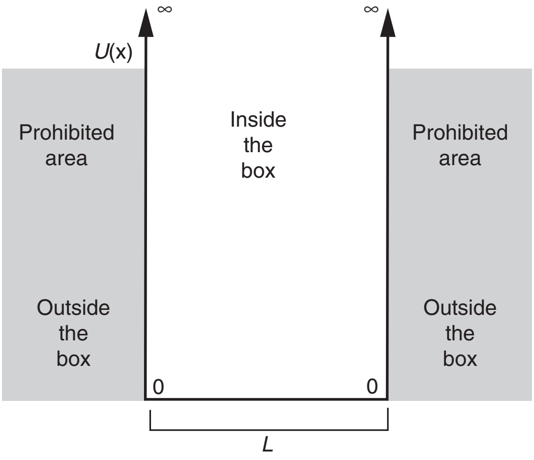


Figure 7 Potential energy of a particle in a box.

For such a region time-independent Schrödinger wave equation can be written as

$$\nabla^2\psi(r,t) + \frac{2\,m(E-V)}{\hbar^2}\psi(r,t) = 0$$

which reduces within the box into

$$\nabla^2\psi(r,t)+\frac{2mE}{\hbar^2}\psi(r,t)=0$$

As the particle is within the 1D box [here $\psi\ (r,t)$ is replaced by $\psi(x,t)$, without any loss of generality, as particle is restricted to move only along x-axis keeping y- and z-axes constant, hence, ∇^2 can be reduced to $\partial^2/\partial x^2$ which in this situation is nothing but d^2/dx^2 then the wave associated with it can be represented by

$$\frac{d^2\psi(x,t)}{dx^2}+\frac{2mE}{\hbar^2}\ \psi(x,t)=0 \Rightarrow \frac{d^2\psi(x,t)}{dx^2}+k^2\psi(x,t)=0 \tag{2.56}$$

where

$$k^2=\frac{2mE}{\hbar^2} \tag{2.57}$$

The general solution of Eq. (2.56) is given by

$$\psi(x)=Ae^{ikx}+Be^{-ikx} \tag{2.58}$$

Since the probability of finding the particle is zero at $x = 0$ and $x = L$. Therefore, $\psi(x) = 0$ at $x = 0$ which gives

$$A+B=0 \Rightarrow A=-B$$

and $\psi(x) = 0$ at $x = L$ which gives

$$Ae^{ikL}+Be^{-ikL}=0 \Rightarrow A(e^{ikL}+e^{-ikL})=0 \quad \Rightarrow 2iA\sin kL=0$$

As $A \neq 0$, therefore,

$$\sin kL=0 \Rightarrow kL=n\pi \quad \Rightarrow k=\frac{n\pi}{L} \tag{2.59}$$

where $n = 1, 2, 3, \ldots$. Putting value of k together with B in Eq. (2.58) we get

$$\psi_n(x)=A\sin\frac{n\pi x}{L} \tag{2.60}$$

2.18 Energy Eigenvalues

From Eqs. (2.57) and (2.59) we have

$$k^2=\frac{2mE}{\hbar^2} \quad \text{and} \quad k^2=\frac{n^2\pi^2}{L^2}$$

which gives

$$\frac{n^2\pi^2}{L^2}=\frac{2mE}{\hbar^2} \quad \text{or} \quad E_n=\frac{n^2\pi^2\hbar^2}{2mL^2} \quad \text{or} \quad E_n=\frac{n^2h^2}{8mL^2} \tag{2.61}$$

(Here we have deliberately introduced the subscript 'n' with energy 'E' in order to denote the energy of different energy levels). Equation (2.61) gives the allowed values of energy or the energy eigenvalues for the particle. From Eq. (2.60), we draw following conclusions:

1. We cannot take principal quantum number $n = 0$ because for $n = 0$ we have $k = 0$ and $E_n = 0$ and hence $\psi = 0$ everywhere in the box. This means that a particle with zero energy cannot be present in the box. In other words, a particle in a box cannot have zero energy.

2. This equation shows that eigenvalues of energy are discrete and not continuous. These values are called energy level of the particle.
3. The lowest energy of the particle is obtained by putting $n = 1$ in Eq. (2.61), that is,

$$E_1 = \frac{h^2}{8mL^2} \quad \text{or} \quad E_n = n^2 E_1$$

4. The spacing between nth energy level and next higher level increases as

$$(n+1)^2 E_1 - n^2 E_1 = (2n+1)E_1$$

Figure 8 shows the energy level diagram of the particle.

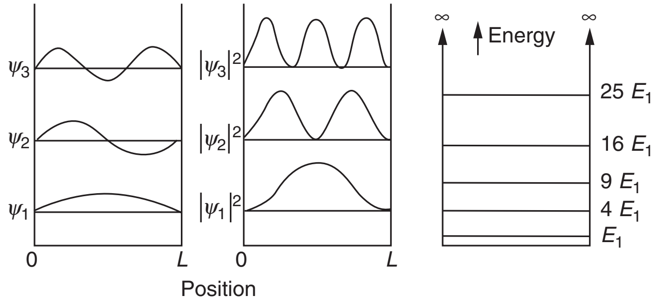


Figure 8 Energy level.

2.19 Eigenfunction (Normalization of Wave Function)

For the region $0 < x < L$, the wave function is given by

$$\psi_n(x) = A \sin \frac{n\pi x}{L}$$

and for the region $x \le 0$ and $x \ge L$, the wave function is given by

$$\psi_n(x) = 0$$

The value of constant 'A' can be found out from the normalization condition. Since the total probability of finding the particle within the box is unity, therefore

$$\int_0^L \psi_n(x)\ \psi_n^*(x)\ dx = 1 \Rightarrow \int_0^L |\psi_n(x)|^2\ dx = 1 \Rightarrow A^2 \int_0^L \sin^2 \frac{n\pi x}{L}\ dx = 1$$

$$\Rightarrow A^2 \int_0^L \frac{1}{2}\left(1 - \cos \frac{2n\pi x}{L}\right) dx = 1 \Rightarrow \frac{A^2}{2}\left[x - \frac{L}{2n\pi} \sin \frac{2n\pi x}{L}\right]_0^L = 1$$

$$\Rightarrow \frac{A^2}{2} L = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$$

Hence the normalized wave function is given by

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

The normalized wave functions for the three states (ψ_1, ψ_2 and ψ_3) together with probability densities $|\psi_1|^2$, $|\psi_2|^2$, $|\psi_3|^2$ are shown in Fig. 8. Although ψ_n may be positive as well as negative, $|\psi_n|^2$ is always positive. Since ψ_n is normalized, its value at a given 'x' is equal to the probability density of finding the particle there. In every case, $|\psi_n|^2 = 0$ at $x = 0$ and $x = L$.

At a particular place in the box the probability of finding the particle may be very different for different quantum number. For instance, $|\psi_1|^2$ has its maximum value of $2/L$ in the middle of the box, while $|\psi_2|^2 = 0$ there. A particle in the lowest energy level of $n = 1$ is most likely to be in the middle of the box, while a particle in the next higher state of $n = 2$ is never there.

Solved Examples

Example 1

Calculate the de-Broglie wavelength associated with a proton moving with a velocity equal to 1/20th the velocity of light. Given mass of proton $m = 1.67 \times 10^{-27}$ kg, $h = 6.62 \times 10^{-34}$ J-s.

Solution: According to the de-Broglie hypothesis

$$\lambda = \frac{h}{mv}$$

We have $m = 1.67 \times 10^{-27}$ kg, $h = 6.62 \times 10^{-34}$ J-s. Now according to the question, it is given that velocity is equal to 1/20th the velocity of light. So

$$v = \frac{1}{20} \times 3 \times 10^8 = 1.5 \times 10^7 \text{ m/s}$$

The wavelength is

$$\lambda = \frac{6.62 \times 10^{-34}}{1.67 \times 10^{-27} \times 1.5 \times 10^7} = 2.643 \times 10^{-14} \text{ m}$$

Example 2

Calculate the de-Broglie wavelength of a neutron having kinetic energy of 12.8 MeV. Mass of electron $m = 1.67 \times 10^{-27}$ kg.

Solution: According to the de-Broglie hypothesis

$$\lambda = \frac{h}{\sqrt{2mE}}$$

We have $h = 6.62 \times 10^{-34}$ J-s, $m = 1.67 \times 10^{-27}$ kg. Also, $E = 12.8 \text{ MeV} = 12.8 \times 10^6 \times 1.6 \times 10^{-19}$ J. The de-Broglie wavelength is

$$\lambda = \frac{6.62 \times 10^{-34}}{\sqrt{2 \times 1.67 \times 10^{-27} \times 12.8 \times 10^6 \times 1.6 \times 10^{-19}}} = 8.02 \times 10^{-15} \text{ m}$$

Example 3

Derive an expression for de-Broglie wavelength of helium atom having energy at temperature T K.

Solution: According to the de-Broglie hypothesis

$$\lambda = \frac{h}{\sqrt{2mE}}$$

From kinetic theory of gases,

$$E = \frac{3}{2}kT$$

where k is the Boltzmann constant. Substituting this value of E in the formula above we get the expression for de-Broglie wavelength as

$$\lambda = \frac{h}{\sqrt{2m \times \frac{3}{2}kT}} = \frac{h}{\sqrt{3mkT}}$$

Example 4

Compare the wavelength of a photon and an electron if the two have the same momentum.

Solution: For photon as well as the electron the de-Broglie wavelength is

$$\lambda = \frac{h}{p}$$

Since momentum of the particles is same, therefore according to the de-Broglie hypothesis

$$\lambda_{\text{ph}} = \lambda_{\text{e}}$$

Example 5

Show that the de-Broglie wavelength of a particle of rest mass m_0, potential V and charge q is given by

$$\lambda = \frac{h}{\sqrt{2m_0qV\left(1 + \frac{qV}{2m_0c^2}\right)}}$$

Solution: When an electron of mass m_0 (rest mass) and having charge 'q' is accelerated through an ordinary
potential V, then non-relativistic kinetic energy is considered. Hence,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_o K}} = \frac{h}{\sqrt{2m_0 qV}}$$

If potential is very high so that relativistic kinetic energy would be considered, then we have

$$E = K + m_0 c^2 \Rightarrow \sqrt{p^2c^2 + m_0^2 c^4} = qV + m_0 c^2$$

Squaring both sides, we get

$$p^2c^2 + m_0^2 c^4 = (qV)^2 + m_0^2 c^4 + 2qVm_0 c^2$$

$$\Rightarrow p = \sqrt{2m_0 qV\left(1 + \frac{qV}{2m_0 c^2}\right)}$$

The de-Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_0 qV\left(1 + \frac{qV}{2m_0 c^2}\right)}}$$

Example 6

An electron has a speed of 600 m/s with an accuracy of 0.005%. Calculate the uncertainty with which we can locate the position of the electron. Given $m = 1.67 \times 10^{-31}$ kg, $h = 6.62 \times 10^{-34}$ J-s.

Solution: By Heisenberg uncertainty principle

$$\Delta x \Delta p \geq \hbar \geq \frac{h}{2\pi}$$

$$\Rightarrow \Delta x \geq \frac{h}{2\pi \Delta p} \geq \frac{h}{2\pi m \Delta v}$$

Now

$$\Delta v = \frac{0.005}{100} \times 600 = 0.03 \text{ m/s}$$

Given $m = 1.67 \times 10^{-31}$ kg, $h = 6.62 \times 10^{-34}$ J-s. So

$$\Delta x \geq \frac{6.6 \times 10^{-34}}{2\pi \times 9.1 \times 10^{-31} \times 0.03} \geq 4.86 \times 10^{-3}\,\text{m}$$

Example 7

A hydrogen atom is having the radius 0.53Å. Estimate the minimum energy an electron can have in this atom.

Solution: We have $\Delta x = 0.53\ \text{Å} = 5.3\ \times 10^{-11}\,\text{m}$. Now

$$\Delta x \Delta p \geq \hbar \geq \frac{h}{2\pi}$$

$$\Rightarrow \Delta p \geq \frac{h}{2\pi \Delta x} \geq \frac{6.6\times 10^{-34}}{2\pi \times 5.3\times 10^{-11}} \geq 19.8\times 10^{-25}\ \text{kg-m/sec}$$

Kinetic energy

$$K = \frac{p^2}{2\,m} = \frac{19.8\times 10^{-25}}{2\times 9.1\times 10^{-31}} = 10.8\times 10^{-19}\ \text{J}$$

Example 8

Calculate the energy difference between the ground state and first excited state for electron if the length of the box is 10^{-8} cm.

Solution: We have

$$E_n = \frac{n^2 h^2}{8mL^2}$$

The lowest energy state corresponds to $n = 1$ and is

$$E_1 = \frac{h^2}{8mL^2} = \frac{6.6\times 10^{-34}\ \text{J-s}}{8\times 9.1\times 10^{-31}\ \text{kg}\times 10^{-10}\ \text{m}} = 0.603\times 10^{-17}\,\text{J}$$

In terms of eV

$$E_1 = \frac{0.603\times 10^{-17}}{1.6\times 10^{-19}}\,\text{eV} = 37.7\ \text{eV}$$

The first excited state corresponds to $n = 2$ and is

$$E_2 = \frac{2^2 h^2}{8mL^2} = 4\,E_1 = 4\times 37.7\ \text{eV} = 150.8\ \text{eV}$$

Now the difference is

$$\Delta E = E_2 - E_1 = 150.8 - 37.7 = 113.1\ \text{eV}$$

Short Answers of Some Important Questions

1. What do you understand by wave-particle duality?

   Answer: Radiation possesses dual nature. Sometimes it behaves as a wave when it interacts with radiation, at other times it behaves as a particle when it interacts with matter.

2. State de-Broglie hypothesis.

   Answer: According to de-Broglie hypothesis, a particle in motion always has a wave associated with it and the motion of the particle is guided by that wave. These waves are called de-Broglie waves or matter waves. The wavelength of matter waves is given by the relation

   $$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mK}}$$

   where m is the mass, v is the ordinary velocity and k is the kinetic energy of the particle.

3. What is the objective of Davisson and Germar experiment?

   Answer: In order to confirm the de-Broglie hypothesis, Davisson and Germer performed an experiment in 1927 by diffraction of electrons. In this experiment, the wavelength of electrons calculated from the diffraction pattern is nearly the same as the value calculated from the de-Broglie hypothesis which confirmed de-Broglie hypothesis.

4. What is wave-packet?

   Answer: When plane waves of different wavelengths travel simultaneously in the same direction along a straight line through a dispersive medium (a medium in which the phase velocity of a wave depends on its wavelength) successive groups of the waves are produced. These wave groups are called wave-packet.

5. What is phase velocity?

   Answer: The velocity of each individual wave of a wave-packet is known as phase velocity. The phase velocity is also called 'wave velocity' or 'velocity of propagation'. It is denoted by v_p and defined as

   $$v_p = \frac{\omega}{k}$$

   where ω is the angular velocity and k is the propagation constant of the wave.

6. What is group velocity?

   Answer: The average velocity through which the wave-packet propagates in the medium is called group velocity (v_g). The group velocity may also be defined as the velocity with which the energy in the group is transmitted. However, the individual waves travel inside the group with their velocities

   $$v_g = \frac{d\omega}{dk}$$

7. State Heisenberg's uncertainty principle.

   Answer: It is impossible to determine the exact position and momentum of a particle simultaneously. If Δx and Δp are the position and momentum of a particle, respectively, then

   $$\Delta x \, \Delta p \geq \hbar$$

8. What is wave function?

   Answer: Schrödinger introduced a mathematical function ψ which is the variable quantity associated with the moving particle and is a complex function of space co-ordinate of particle and time. ψ is known as wave function.

9. What is normalized wave function?

   Answer: If the total probability of finding the particle anywhere in the space is 1, then the corresponding wave function will be normalized wave function.

10. What are eigenvalues and eigenfunctions?

    Answer: When the Schrödinger time-independent wave equation is solved for any particle using its wave function under certain boundary conditions, the solutions exist only for particular values of energy E_n which are known as the eigenvalues. The corresponding wave function ψ_n to the eigenvalue is called eigenfunction.

Important Points and Formulas

1. According to dual nature of light, radiation sometimes behaves as particle and sometimes as wave. Both particle and wave properties of radiation cannot appear simultaneously.
2. A particle is specified by mass, velocity, energy and momentum; however, a wave is specified by frequency, wavelength, amplitude and intensity.
3. A particle in motion always has a wave associated with it and the motion of the particle is guided by that wave. These waves are called de-Broglie waves or matter waves.
4. The wavelength λ of matter waves associated with a moving particle is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mK}}$$

5. In order to confirm the de-Broglie hypothesis, Davisson and Germer performed an experiment in 1927 to observe the diffraction of electrons. From this experiment the wavelength of the diffracted electrons calculated from the diffraction pattern is nearly same as the value calculated from the de-Broglie hypothesis.
6. The expressions for group velocity and phase velocity are

$$v_g = \frac{d\omega}{dk} \text{ and } v_p = \frac{\omega}{k}$$

7. Heisenberg's uncertainty principle can be defined as follows: "It is impossible to determine the exact position and momentum of a particle simultaneously."
8. Schrodinger time-independent wave equation is

$$\nabla^2\psi(r,t) + \frac{8\pi^2 m(E-V)}{h^2}\psi(r,t) = 0$$

9. Schrodinger time-dependent wave equation is

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\psi(r,t) = i\hbar\frac{\partial\psi(r,t)}{\partial t}$$

10. Energy eigenvalue $E_n = \dfrac{n^2h^2}{8mL^2}$

Multiple Choice Questions

1. The idea of dual nature of light for all microscopic particle was given by
   (a) Einstein **(b)** Planck
   (c) de-Broglie **(d)** None of these
2. Which of the following phenomena can be expressed by particle nature of light?
   (a) Photoelectric effect
   (b) Interference
   (c) Diffraction
   (d) Polarization
3. Which of the following phenomena can be expressed on the basis of wave nature of light?
   (a) Interference **(b)** Photoelectric effect
   (c) Compton effect **(d)** None of these
4. Which of the following phenomena can be expressed on the basis of both wave nature and particle nature of light?
   (a) Diffraction
   (b) Interference
   (c) Polarization
   (d) Refraction and reflection
5. The angular momentum of an electron in its orbit is given by
   (a) $\frac{h}{3\pi}$ **(b)** $\frac{nh}{2\pi}$
   (c) $\frac{3h}{2\pi}$ **(d)** $\frac{4h}{2\pi}$
6. A material particle in motion is equivalent to
   (a) single wave **(b)** group of waves
   (c) straight line **(d)** None of these
7. Which of the following is particle property?
   (a) Wavelength **(b)** Frequency
   (c) Momentum **(d)** None of these

8. The matter wave associated with a particle of mass *m* and moving with velocity *v* is

(a) $\lambda = \frac{h^2}{mv^2}$ **(b)** $\lambda = \frac{h}{mv}$

(c) $\lambda = \frac{h}{mv}$ **(d)** $\lambda = \frac{(mv)^2}{h}$

9. The wavelength of a wave associated with an electron accelerated through 150 V is

(a) 1.38 Å **(b)** $\frac{12.27}{\sqrt{V}}$

(c) $\frac{12.85}{V}$ **(d)** $\frac{12.72}{V^{3/2}}$

10. Davison and Germer experiment relates to

(a) electron diffraction
(b) electron interference
(c) polarization
(d) None of these

11. Heisenberg's uncertainty is

(a) $\Delta x \cdot \Delta p \geq \frac{h}{2}$ **(b)** $\Delta x \cdot \Delta p = h^2$

(c) $\Delta x \cdot \Delta p = h$ **(d)** $\Delta E \cdot \Delta p = h$

12. Uncertainty principle was discovered by

(a) Thomson **(b)** Heisenberg
(c) Einstein **(d)** de-Broglie

13. The relation between v_{p} and v_{g} is

(a) $v_{\mathrm{g}} = v_{\mathrm{p}} - \lambda \frac{dv_{\mathrm{p}}}{d\lambda}$ **(b)** $v_{\mathrm{g}} = v_{\mathrm{p}} + \lambda \frac{dv_{\mathrm{p}}}{d\lambda}$

(c) $v_{\mathrm{g}} = v_{\mathrm{p}}$ **(d)** None of these

14. For non-relativistic motion of a particle, the phase velocity will be

(a) v **(b)** $v/2$

(c) $\frac{v^3}{c}$ **(d)** c

15. For relativistic motion of a particle, the phase velocity will be

(a) c **(b)** $v/2$

(c) $\frac{c^2}{v}$ **(d)** $\frac{v^2}{c}$

16. For relativistic kinetic energy, the de-Broglie wavelength λ is

(a) $\frac{hc}{\sqrt{k(k - 2m_0c^2)}}$ **(b)** $\frac{hc}{\sqrt{k(k + 2m_0c^2)}}$

(c) $\sqrt{\frac{hc}{k(k - 2m_0c^2)}}$ **(d)** None of these

17. The uncertainty principle is practically applicable to

(a) macroscopic system
(b) microscopic systems
(c) Both (a) and (b)
(d) Neither microscopic nor macroscopic

18. Which of the following is canonically conjugate quantity?

(a) ΔE and Δm **(b)** Δm and ΔE

(c) ΔE and Δt **(d)** Δt and Δl

19. The product of wave function ψ and its complex conjugate is known as

(a) probability density
(b) charge density
(c) mass density
(d) current density

20. The normalized eigenfunction for one-dimension potential box of length ‘*l*’ is

(a) $\sqrt{\frac{2}{l}} \sin \frac{n\pi l}{x}$ **(b)** $\sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}$

(c) $\sqrt{\frac{l}{2}} \sin \frac{n\pi l}{x}$ **(d)** $\sqrt{\frac{l}{2}} \sin \frac{n\pi x}{l}$

21. The eigenfunction for the particle in one-dimension potential box of length *l* is

(a) $E_n = \frac{n^2h^2}{8ml^2}$ **(b)** $\frac{n^2h^2}{16ml^2}$

(c) $\frac{8ml^2}{n^2h^2}$ **(d)** None of these

22. Schrodinger wave equation is applicable to

(a) non-relativistic motion
(b) relativistic motion
(c) Neither for (a) nor (b)
(d) Both (a) and (b)

Short Answer Type Questions

1. What is dual nature of light?
2. What are matter waves?
3. What do you understand by de-Broglie wavelength?
4. What is wave packet?
5. What do you understand by uncertainty principle?
6. Give the relation between phase velocity and group velocity.
7. What do you understand by Schrödinger wave equation?
8. What is eigenfunction?
9. Write down the Schrödinger time-independent wave equation.
10. Give the property of wave function.

Long Answer Type Questions

1. What are matter waves? Explain why a single wave cannot resemble a material particle.
2. Explain de-Broglie hypothesis and discuss how it is experimentally verified.
3. A photon of frequency ν is scattered by an electron of rest mass m_0. The scattered photon of frequency ν' travels in a direction inclined at 90° with the initial direction. Prove that the de-Broglie wavelength of recoil electron is given by
$$\lambda = \frac{c}{\sqrt{\nu^2 + \nu'^2}}$$
4. Establish the following relations:
   (a) $v_g = v_p - \lambda(dv_p/d\lambda)$
   (b) $v_g \cdot v_p = c^2$
5. Show that the phase velocity of a de-Broglie wave of wavelength λ is given by
$$v_p = c\sqrt{1 + \left(\frac{m_0^2 c^2}{h^2}\right)\lambda^2}$$
6. State Heisenberg's uncertainty principle. Discuss its significance and importance.
7. Explain Heisenberg's uncertainty principle. Discuss one application of this principle.
8. What is uncertainty principle? Apply it to prove the non-existence of electron and existence of proton in the nucleus.
9. What is physical significance of wave function? Derive time-independent Schrödinger wave equation. What happens if the particle is free?
10. Derive time-dependent Schrödinger wave equation. What is the physical significance of wave function used in this equation?
11. Find an expression for eigenvalue and eigenfunction of a particle in one-dimensional potential box.

Numerical Problems

1. An electron and photon each has λ = 2 Å. Compare their (a) momenta, (b) total energy and (c) ratio of kinetic energy.
2. What is the de-Broglie wavelength of an electron accelerated from rest through a potential difference of 100 V?

3. How fast should an electron move in order to have its wavelength equal to the distance travelled by it in one second?

4. Two particles A and B are in motion. If the wavelength associated with particle A is 5×10^{-8}m, calculate the wavelength of particle B if its momentum is half of that of A.

5. Show that the wavelength associated with an electron becomes equal to the wavelength associated with a proton if velocity of electron is 1836 times velocity of proton.

6. The sodium light has a characteristic yellow color due to wavelength 5890 Å. Find the mass equivalence of this photon.

7. Calculate de-Broglie wavelength associated with nitrogen at 3.0 atm. pressure and 27°C. Mass of N_2 molecule 4.65×10^{-26} kg.

8. Calculate the wavelength associated with a (a) 1 MeV electron, (b) 1 MeV proton and (c) 1 MeV photon (KE).

Answers

Multiple Choice Questions

1. (c)
2. (a)
3. (a)
4. (d)
5. (b)
6. (b)
7. (c)
8. (b)
9. (b)
10. (a)
11. (a)
12. (b)
13. (a)
14. (b)
15. (c)
16. (b)
17. (b)
18. (c)
19. (a)
20. (b)
21. (a)
22. (a)

Numerical Problems

1. (a) Momentum is equal (3.31×10^{-24} kg-m/sec);
(b) 0.51 MeV, 6.21 KeV;
(c) 6.05×10^{-3}

2. 1.226 Å

3. 2.697×10^{-2} m/sec

4. 10×10^{-8} m

6. 3.74×10^{-36} kg

7. 0.628 Å

8. (a) 8.75×10^{-3} Å,
(b) 2.875×10^{-4} Å,
(c) 1.24×10^{-2} Å

3 Wave Optics: Interference

LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Interference of light.
- Interference in thin films (parallel and wedge shaped film).
- Newton's rings.

3.1 Introduction

Optics is the branch of physics in which we study the nature of light and the phenomenon exhibited by it. Before 19th century many theories have been put forward in this direction. The most famous among them were Newton's corpuscular theory, Huygen's wave theory, Maxwell's electromagnetic theory, Planck quantum theory of light, etc.

According to corpuscular theory, light consists of very small, weightless and perfectly elastic particles called corpuscles. Reflection, refraction and rectilinear propagation can be explained by this theory but it fails to explain interference, diffraction and polarization.

Huygen's suggested that light creates periodic disturbance which travels as waves in a manner very similar to that of sound waves. It explains reflection, refraction, interference and diffraction but fails to explain polarization and rectilinear propagation of light.

Thereafter, Maxwell's considered the light to be electromagnetic in nature. Therefore, no material medium is required for its propagation. In this reference, Planck assumed that light consists of small particles in the form of discrete bundles of energy called quanta or photons and the energy of one photon is equal to $h\upsilon$ where h is Planck constant and υ is frequency.

Presently, it is assumed that light behaves in a dual nature: one is particle and other wave nature. Therefore, on the basis of wave nature we can explain the phenomena of interference, diffraction and polarization (also called wave optics). However, photoelectric effect and Compton effect can be explained by particle nature. In this section, we limit ourselves to the wave nature of light.

3.2 Interference of Light

When two waves of same frequency and constant phase difference travel simultaneously in the same direction, then there is a change in the intensity of the waves due to superposition of two waves. This change in the intensity is said to be interference.

The points where change in intensity is greater than that of the sum of the intensities due to the individual waves are called constructive interference whereas some other points where change in intensities is less than that of the sum of the intensities due to individual waves are called destructive interference.

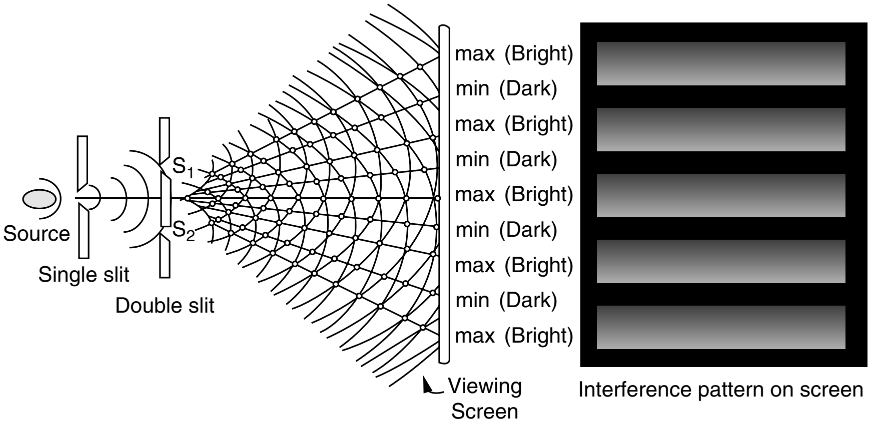


Figure 1 Fringe pattern.

In other words, one can say that interference is nothing but the redistribution of energy. After interference, we get interference fringes which are alternate dark and bright bands of regular or irregular shape (see Fig. 1).

3.3 Superposition

We have already discussed in the previous section that when two waves travel in a medium, there is a modification in intensity of the waves due to superposition of two waves. As a result a new wave is formed whose amplitude is determined by superposition principle. Let A be the resultant amplitude and A_1, A_2, A_3, A_4, … be the amplitudes of the individual waves. Then

$$A = A_1 \pm A_2 \pm A_3 \pm A_4 \pm \cdots$$

where '+ve' sign stands for amplitude of waves in same direction whereas '–ve' sign stands for opposite direction. Thus, *the resultant amplitude at a point and at any instant of time is the algebraic sum of the amplitudes of the individual waves*. This is known as a principle of superposition.

3.4 Types of Interference

1. **Division of wave fronts:** In this case, the wave front is divided into two parts to produce interference fringes. For example, laser, Fresnel biprism, Young's double slit experiment, Fresnel mirrors, etc.
2. **Division of amplitude:** Under this category, amplitude of the incident light is divided into two parts due to reflection or refraction to produce interference fringes. For example, thin films, Newton's ring, Michelson interferometer, etc.

3.5 Theory of Interference

To understand the phenomena of interference and derivation of an expression for the change in intensity at any point on the screen, consider a monochromatic light source S emitting waves of wavelength λ. S_1 and S_2 are two narrow slits close together and equidistant from S as shown in Fig. 2.

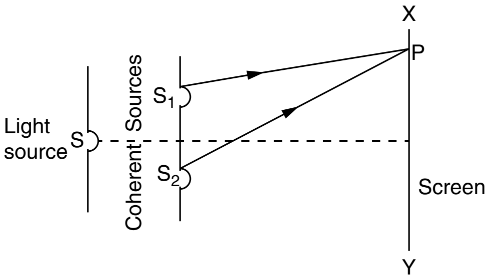


Figure 2 Theory of interference.

Let a and b be amplitudes of two waves at P from S_1 and S_2, respectively and y_1 and y_2 be the corresponding displacements. Then

$$y_1 = a\sin(\omega t) \tag{3.1}$$

$$y_2 = a\sin(\omega t + \delta) \tag{3.2}$$

where δ is the phase difference between the two waves. Using the phenomena of superposition, the resultant displacement is

$$\begin{aligned} y &= y_1 + y_2 \\ &= a\sin(\omega t) + b\sin(\omega t + \delta) \\ &= a\sin(\omega t) + b\sin\omega t\cos\delta + b\cos\omega t\sin\delta \\ &= (a + b\cos\delta)\sin\omega t + b\cos\omega t\sin\delta \end{aligned} \tag{3.3}$$

Let

$$(a + b\cos\delta) = R\cos\theta \tag{3.4}$$

$$b\sin\delta = R\sin\theta \tag{3.5}$$

where R and δ are the new constants. Now substituting the values of Eqs. (3.4) and (3.5) in Eq. (3.3) we get

$$y = \sin\omega t R\cos\theta + \cos\omega t R\sin\theta = R\,\sin(\omega t + \theta) \tag{3.6}$$

Thus, the resultant displacement at point P on screen is simple harmonic of amplitude R and phase θ. Squaring and adding Eqs. (3.4) and (3.5) we get

$$\begin{gathered} R^2\cos^2\theta + R^2\sin^2\theta = (a + b\cos\delta)^2 + (b\sin\delta)^2 \\ \Rightarrow R^2 = a^2 + b^2\cos^2\delta + 2ab\cos\delta + b^2\sin^2\delta = a^2 + b^2 + 2ab\cos\delta \end{gathered} \tag{3.7}$$

Since, we know that intensity I is proportional to the square of the amplitude R^2. So,

$$I = R^2 = a^2 + b^2 + 2ab\cos\delta \tag{3.8}$$

Thus from Eq. (3.8) we can conclude that resultant intensity is different than the sum of the intensities due to individual waves.

3.5.1 Constructive Interference or Maxima

We know that maximum and minimum value of cosine is +1 and −1 respectively. So

$$\cos\delta = +1 \quad \text{or} \quad \delta = 0, 2\pi, 4\pi, 6\pi, \ldots$$

Generally, we can write

$$\delta = 2n\pi, \quad n = 0, 1, 2, 3, \ldots$$

Since

$$\text{Path difference} = \frac{\lambda}{2\pi}\ \text{Phase difference}$$

therefore we have

$$\text{Path difference} = \frac{\lambda}{2\pi} \times 2n\pi = n\lambda \tag{3.9}$$

Thus, the path difference between two interfering waves is equal to integral multiples of λ.
Putting the value of $\cos\delta = +1$ in Eq. (3.8) we get

$$I_{\max} = R_{\max}^2 = a^2 + b^2 + 2ab = (a+b)^2 \tag{3.10}$$

Equation (3.10) clearly indicates that the maximum intensity is greater than the sum of the intensities due to two individual waves.

3.5.2 Destructive Interference or Minima

We know that

$$\cos\delta = -1 \quad \text{or} \quad \delta = \pi, 3\pi, 5\pi, \ldots$$

Generally, we can write

$$\delta = (2n-1)\,\pi, \quad n = 1, 2, 3, \ldots$$

or

$$\delta = (2n+1)\,\pi, \quad n = 0, 1, 2, 3, \ldots$$

Since

$$\text{Path difference} = \frac{\lambda}{2\pi}\ \text{Phase difference}$$

Therefore

$$\text{Path difference} = \frac{\lambda}{2\pi} \times (2n-1)\,\pi = (2n-1)\frac{\lambda}{2}$$

or

$$= \frac{\lambda}{2\pi} \times (2n+1)\,\pi = (2n+1)\frac{\lambda}{2} \tag{3.11}$$

Thus, path difference between two interfering waves is equal to odd multiples of $\lambda/2$. Putting the value of $\cos\delta = -1$ in Eq. (3.8) we get

$$I_{\min} = R^2{}_{\min} = a^2 + b^2 - 2ab = (a-b)^2 \tag{3.12}$$

Equation (3.12) clearly indicates that the minimum intensity is less than the sum of the intensities due to the individual intensities.

Let us see what happens if the amplitude of two waves is same, that is, $a = b$:

$$I_{\max} = a^2 + a^2 + 2aa = 4a^2 \tag{3.13}$$

$$I_{\min} = a^2 + a^2 - 2aa = 0 \tag{3.14}$$

From Eq. (3.13) one can conclude that the resultant intensity is maximum or brightness occurs at points on which $\delta = 2n\pi$ and is $4a^2$; however, it is minimum or darkness appears at points on which $\delta = (2n-1)\pi$ or $(2n+1)\pi$ and is zero. We plot the curve between maximum and minimum intensities with phase difference δ. The intensity varies between zero and $4a^2$ depending upon the phase difference δ between two interfering waves as shown in Fig. 3; it is called the *intensity distribution curve*. The law of conservation of energy is also true in the formation of interference fringes.

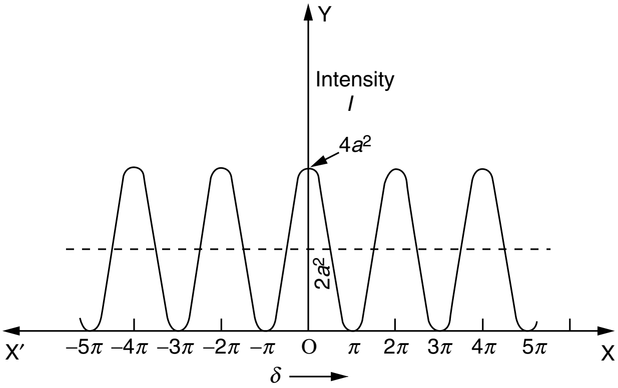


Figure 3 Variation of intensity with δ.

Average Intensity: It is the average of the minimum and maximum intensities and it is given by

$$I_{av} = \frac{\int_0^{2\pi} I\,d\delta}{\int_0^{2\pi} d\delta} = \frac{\int_0^{2\pi} (a^2 + b^2 + 2ab\cos\delta)\,d\delta}{\int_0^{2\pi} d\delta} = \frac{[a^2\delta + b^2\delta + 2ab\sin\delta]_0^{2\pi}}{[\delta]_0^{2\pi}} = \frac{[a^2 + b^2]\,2\pi}{2\pi} = 2a^2 \quad (\because a = b)$$

The average intensity is equal to the sum of the separate intensities, that is, energy is neither created nor destroyed but it merely redistributes in the interference pattern. Thus, we prove that the phenomenon of interference is in accordance with the law of conservation of energy.

3.6 Coherent Sources

Coherent sources are nothing but two light waves of same frequency of wavelength having same amplitude and always a constant phase difference between them. In actual practice two independent sources cannot be coherent because they cannot maintain a constant phase difference between them. But for experimental purposes two virtual sources obtained from a single source can act as coherent.

3.6.1 Condition for the Interference or Permanent or Sustained Interference

The conditions for sustained interference are as follows:

1. The first and foremost condition is that the two interfering sources must be coherent, that is, they always should maintain constant phase difference.
2. The wavelength and time period of the two interfering sources must be the same.
3. The amplitude or intensities must be equal or very nearly equal.
4. The separation between the two coherent sources must be as small as possible.
5. Two sources should be narrow.
6. The distance between two sources and screen should be as large as possible.

3.7 Fringe Width

Let us consider S to be the source illuminated with monochromatic light having wavelength λ (see Fig. 4). Let S_1 and S_2 be the two equidistant coherent sources from S. Let $2d$ be the separation between two narrow slits S_1 and S_2 and D be the distance of screen from two coherent sources. Let us consider P to be the point on the screen at a distance x from O where bright or dark bands are located. Thus

$$\text{Path difference} = S_2P - S_1P$$

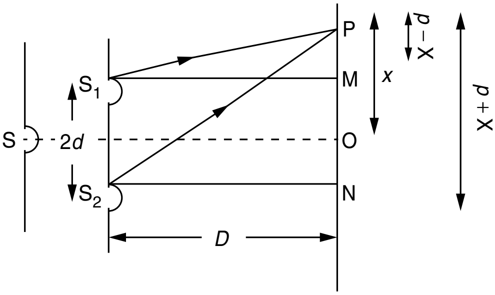


Figure 4 Young double slit experiment.

To determine the values of S_2P and S_1P, we consider the triangles S_2NP and S_1MP in Fig. 4. Now

$$\begin{aligned} S_2P - S_1P &= \sqrt{[(S_2N)^2 + (NP)^2} - \sqrt{(S_1M)^2 + (MP)^2]} \\ &= \sqrt{[(D)^2 + (x+d)^2} - \sqrt{(D)^2 + (x-d)^2]} \\ &= D\left(1 + \frac{(x+d)^2}{D^2}\right)^{1/2} - D\left(1 + \frac{(x-d)^2}{D^2}\right)^{1/2} \end{aligned}$$

Using Binomial theorem and neglecting higher terms, we get

$$S_2P - S_1P = D\left(1 + \frac{(x+d)^2}{2D^2}\right) - D\left(1 + \frac{(x-d)^2}{2D^2}\right) = \frac{1}{2D}[(x+d)^2 - (x-d)^2] = \frac{2xd}{D} \tag{3.15}$$

If δ is the corresponding phase difference, then

$$\delta = \frac{2\pi}{\lambda} \times \frac{2xd}{D}$$

3.7.1 Bright Fringe or Maxima

For maxima, path difference should be $n\lambda$. Therefore

$$\frac{2xd}{D} = n\lambda \Rightarrow x = \frac{nD\lambda}{2d}$$

If x_n is the position of nth bright fringe, then

$$x_n = \frac{nD\lambda}{2d}, \quad n = 0, 1, 2, 3, \ldots$$

Fringe width: It is defined as the separation between two consecutive bright fringes. Hence, fringe width (ω) is given by

$$\omega = x_n - x_{n-1} = \frac{nD\lambda}{2d} - \frac{(n-1)D\lambda}{2d} = \frac{D\lambda}{2d} \tag{3.16}$$

3.7.2 Dark Fringe or Minima

For minima, path difference should be $(2n-1)\lambda/2$. Therefore

$$\frac{2xd}{D} = (2n-1)\frac{\lambda}{2} \quad \Rightarrow x = \frac{(2n-1)\,D\lambda}{4d}$$

If x_n is the position of the nth dark fringe, then

$$x_n = \frac{(2n-1)\,D\lambda}{4d}, \quad n = 1, 2, 3, \ldots$$

Fringe width: It is defined as the separation between two consecutive dark fringes. Hence, fringe width (ω) is given by

$$\omega = x_n - x_{n-1} = \frac{(2n-1)\,D\lambda}{4d} - \frac{[2(n-1)-1]\,D\lambda}{4d} = \frac{D\lambda}{2d} \tag{3.17}$$

From Eqs. (3.16) and (3.17), it is clear that fringe width varies directly with D and λ, and inversely with $2d$. We can also conclude that all bright and dark fringes are of equal width.

3.8 Interference in Thin Films

We know that when white light falls on a thin film of oil spread on the surface of water, beautiful colors are seen. Similar colors are also produced by the thin film of soap bubble. This phenomenon can be explained on the basis of interference in thin films.

3.8.1 Interference in Thin Film Due to Reflected Light

Consider a thin film of refractive index μ and thickness t (see Fig. 5). Let a ray SA fall on the upper surface of the film at incident angle i. The ray is partly reflected along AE and partly refracted along AB at angle r. Lower surface also reflects the ray along BC and finally, the ray emerges out from the upper surface of the film along CD. To evaluate the path difference between AE and CD, we draw perpendiculars CF and AG on AE and BC, respectively.

The optical path difference between AE and CD is

$$\Delta = \mu(\mathrm{AB} + \mathrm{BC}) - \mathrm{AF} \tag{3.18}$$

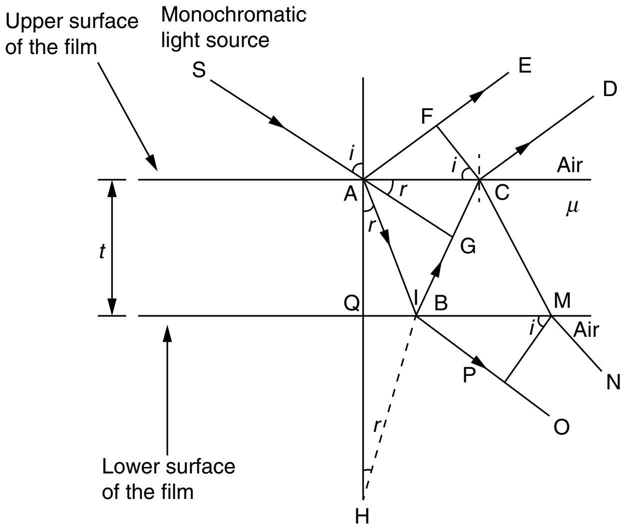


Figure 5 Thin film interference.

From Snell's law, in triangles AFC and AGC

$$\mu = \frac{\sin i}{\sin r} = \frac{\mathrm{AF/AC}}{\mathrm{CG/AC}} = \frac{\mathrm{AF}}{\mathrm{CG}} \tag{3.19}$$

therefore we have

$$\mathrm{AF} = \mu\,\mathrm{CG} \tag{3.20}$$

Putting the value of AF from Eq. (3.20) in Eq. (3.18), we have

$$\begin{aligned}\Delta &= \mu\,(\mathrm{AB} + \mathrm{BC}) - \mu\,\mathrm{CG}\\ &= \mu\,(\mathrm{AB} + \mathrm{BC} - \mathrm{CG}) = \mu\,(\mathrm{HC} - \mathrm{CG})\\ &= \mu\,\mathrm{HG}\end{aligned} \tag{3.21}$$

In the triangle AGH,

$$\cos r = \frac{\mathrm{HG}}{\mathrm{AH}} \quad \Rightarrow HG = \mathrm{AH}\cos r = 2t\cos r$$

(Since triangle AQB congruent triangle BQH, hence AQ = QH = $2t$.) Now $\Delta = 2\mu t\cos r$. A phase change of π equivalent to a path difference of $\lambda/2$ is produced when a ray of light is reflected from the denser medium (Stokes' theorem). Therefore, the effective path difference in this case is

$$\Delta' = 2\mu t\cos r - \frac{\lambda}{2} \tag{3.22}$$

3.8.1.1 Condition for Constructive Interference or Maximum Intensities or Brightness

If the path difference $\Delta' = n\lambda$ where $n = 0, 1, 2, 3, \ldots$ then constructive interference takes place and the film appears bright in the reflected light:

$$2\mu t\cos r - \frac{\lambda}{2} = n\lambda \quad \Rightarrow \quad 2\mu t\cos r = (2n+1)\frac{\lambda}{2} \tag{3.23}$$

3.8.1.2 Condition for Destructive Interference or Minimum Intensities or Darkness

If the path difference $\Delta' = (2n+1)\lambda/2$, where $n = 0, 1, 2, 3, \ldots$, then destructive interference takes place and the film appears dark in the reflected light:

$$2\mu t\cos r - \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2} \quad \Rightarrow \quad 2\mu t\cos r = (n+1)\lambda$$

Since n is an integer, therefore $(n + 1)$ can also be taken as n. Thus

$$2\mu t\cos r = n\lambda \tag{3.24}$$

3.8.2 Interference in Thin Film Due to Reflected Light

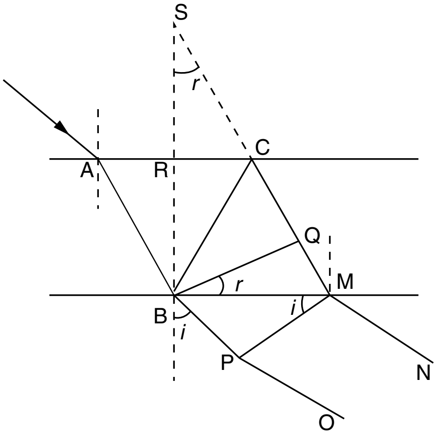


Figure 6 Interference in thin film due to transmitted light

The interference in thin film due to transmitted light is takes place between the ray BO and MN. The optical path difference between the transmitted light BO and MN as shown in Fig. 6 is

$$\Delta = \mu\,(\text{BC} + \text{CM}) - \text{BP}$$

Draw a normal from M on BO at P and from B on CM at Q. From Δ BPM and Δ BQM by Snell's law

$$\mu = \frac{\sin i}{\sin r} = \frac{\text{BP/BM}}{\text{QM/BM}} = \frac{\text{BP}}{\text{QM}}$$

or $$\text{BP} = \mu\text{QM}$$

Then

$$\Delta = \mu(\text{BC} + \text{CM}) - \mu\text{QM} = \mu(\text{BC} + \text{CM} - \text{QM})$$

Since Δ BRC and RSC are congruent, therefore

$$\text{BC} = \text{CS} \quad \text{and} \quad \text{BR} = \text{RS}$$

So

$$\Delta = \mu(\text{SM} - \text{QM}) = \mu\text{SQ}$$

Now

$$\text{SQ} = \text{BS} \cos r = 2t \cos r$$

Hence

$$\Delta = 2\mu t \cos r \tag{3.25}$$

Since no light is reflected from rare-denser medium, hence no additional path difference appears as in reflected case.

3.8.2.1 Condition for Constructive Interference or Maximum Intensities or Brightness

If the path difference $\Delta = n\lambda$ where $n = 0, 1, 2, 3, \ldots$, then constructive interference takes place and the film appears bright in the transmitted light:

$$2\mu t \cos r = n\lambda \tag{3.26}$$

3.8.2.2 Condition for Destructive Interference or Minimum Intensities or Darkness

If the path difference $\Delta = (2n+1)\lambda/2$ where $n = 0, 1, 2, 3, \ldots$, then destructive interference takes place and the film appears dark in the transmitted light:

$$2\mu t \cos r = (2n+1)\frac{\lambda}{2} \tag{3.27}$$

Important Points to Remember

1. If the film is very thin, that is, $t \to 0$ then the net path difference is $\lambda/2$ which is the condition of destructive interference and the film will appear dark.
2. Equations (3.23), (3.24), (3.26) and (3.27) represent that the conditions for maxima and minima in the reflected light are just reverse of the transmitted light. Hence, interference in thin film due to reflected light will be complementary to that observed in transmitted light.

3.9 Colors of Thin Films

If a thin film is exposed with white light, the light will not contain the color whose wavelength satisfies the equation $2\mu t \cos r = n\lambda$ in the reflected system. Therefore, the film will appear colored depending upon the thickness and the angle of inclination. For example, if r and t are constants, the color will be uniform, whereas if r and t vary then different colors are seen (in the case of oil on the water).

3.10 Interference in Thin Film Due to Wedge-Shaped or Thin Film Interference of Increasing Thickness

If there are two surfaces inclined at angle θ at one end and the other end having regularly varying thickness, then it is known as wedge-shaped film. Consider a wedge-shaped film of refractive index μ enclosed by two plane surfaces OA and OB at an angle θ. The thickness of the film gradually increases when we move from O to A (see Fig. 7).

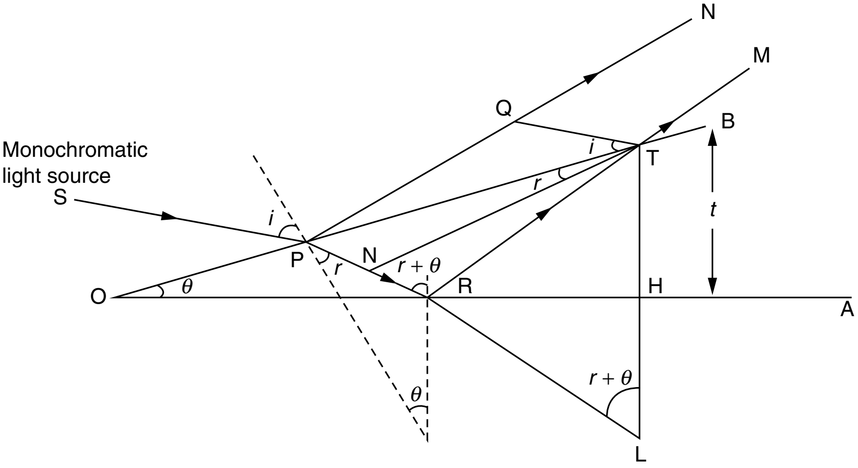


Figure 7 Wedge-shaped film.

Let SP be the incident ray of monochromatic light on the upper surface of the film, PN be the reflected ray and PR be the refracted ray at angle r as shown in Fig. 7. The lower surface also reflects the ray along RT and finally, the ray emerges out from the upper surface of the film along TM. To evaluate the path difference between PN and TM, we draw perpendiculars TQ and TN on PN and PR, respectively. The optical path difference between PN and TM is therefore given as

$$\Delta = \mu(\text{PR} + \text{RT}) - \text{PQ} = \mu(\text{PN} + \text{NR} + \text{RT}) - \text{PQ} \tag{3.28}$$

From simple geometry it may be shown that $\angle\text{PTK} = i$ and $\angle\text{PTN} = r$. Now

$$\mu = \frac{\sin i}{\sin r} = \frac{\text{PQ/PT}}{\text{PN/PT}} = \frac{\text{PQ}}{\text{PN}}$$

$$\text{PQ} = \mu\text{PN} \tag{3.29}$$

Putting the value of PQ from Eq. (3.29) into Eq. (3.28), we have

$$\Delta = \mu(\text{PN} + \text{NR} + \text{RT}) - \mu\text{PN} = \mu(\text{NR} + \text{RT})$$

Now triangles RTH and RHL are congruent. So

$$\text{TH} = \text{HL} = t \text{ and RT} = \text{RL} \tag{3.30}$$

Then

$$\Delta = \mu(\text{NR} + \text{RL}) = \mu(\text{NL}) \tag{3.31}$$

From right-angled triangle TNL we have

$$\frac{\text{NL}}{\text{TL}} = \cos(r+\theta) \Rightarrow \text{NL} = \text{TL}\cos(r+\theta)$$

$$\Rightarrow \text{NL} = 2\text{TH}\cos(r+\theta) \Rightarrow \text{NL} = 2t\cos(r+\theta) \tag{3.32}$$

From Eqs. (3.31) and (3.32), we have

$$\Delta = 2\mu t \cos(r+\theta)$$

Since the ray PN is reflected from a denser medium, therefore a phase change π or path difference $\lambda/2$ is produced. Hence, the effective path difference is

$$\Delta' = 2\mu t \cos(r+\theta) - \frac{\lambda}{2}$$

3.10.1 Condition for Constructive Interference or Maximum Intensities or Brightness

If the path difference $\Delta' = n\lambda$ where $n = 0, 1, 2, 3, \ldots$, then constructive interference takes place and the film appears bright in the reflected light:

$$2\mu t \cos(r+\theta) - \frac{\lambda}{2} = n\lambda \Rightarrow 2\mu t \cos(r+\theta) = (2n+1)\frac{\lambda}{2} \tag{3.33}$$

3.10.2 Condition for Destructive Interference or Minimum Intensities or Darkness

If the path difference $\Delta' = (2n+1)\lambda/2$ where $n = 0, 1, 2, 3, \ldots$, then destructive interference takes place and the film appears dark in the reflected light:

$$2\mu t \cos(r+\theta) - \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2} \Rightarrow 2\mu t \cos(r+\theta) = (n+1)\lambda$$

Since n is an integer, therefore $(n + 1)$ can also be taken as n. Thus

$$2\mu t \cos(r+\theta) = n\lambda \tag{3.34}$$

3.11 Fringe Width

Let the fringes be obtained at a distance x_n from the edge as shown in Fig. 8. Then

$$t = x_n \tan\theta$$

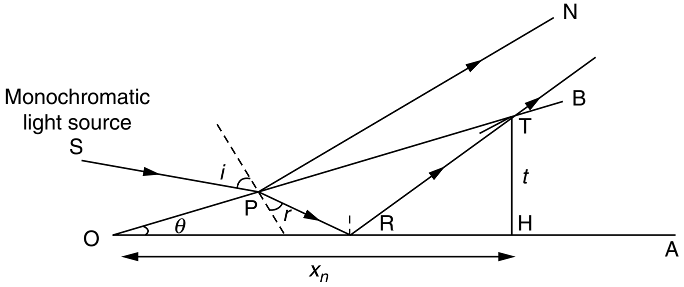


Figure 8 Fringe width in wedge-shaped film.

For small value of θ,

$$\tan\theta = \theta \quad \text{and} \quad \cos(r+\theta) = 1$$

Hence from Eq. (3.34), we get

$$2\mu x_n \theta = n\lambda \tag{3.35}$$

Similarly, if $(n + 1)$ dark fringe is obtained at distance x_{n+1} then

$$2\mu x_{n+1}\theta = (n+1)\lambda \tag{3.36}$$

From Eqs. (3.35) and (3.36), we get

$$2\mu\theta\,(x_{n+1} - x_n) = \lambda$$

So

Fringe width (separation between two consecutive bright or dark fringes) = $x_{n+1} - x_n = \dfrac{\lambda}{2\mu\theta_n}$

Hence fringe width decreases with increase in wedge angle θ.

3.12 Newton Rings

When a plano-convex lens of large radius of curvature is placed on a plane glass plate, a thin wedge-shaped air film is developed between the lower surface of the lens and the upper surface of the plane glass plate. The thickness of the film is zero at the point of contact and gradually increases as we move away from the centre or the point of contact. When a monochromatic light falls normally on the air film, we get alternatively bright and dark circular rings around the point of contact. *The fringes are circular because the air film developed between the lower surface of plano-convex lens and plane glass plate is symmetrical about the point of contact.* This was first investigated by Newton and hence they are called Newton's rings.

3.12.1 Experimental Arrangement

Let S be an extended source of monochromatic light placed at the focus of lens L_1. The light rays emitted from S are made parallel by a lens L_1. The parallel rays fall on a glass plate G inclined at 45° to the parallel rays. The light reflected from the glass plate falls normally on the air film enclosed between the upper and lower surfaces of the film. The light rays reflected from upper surface of the thin film interfere with light reflected from lower surface of the film. The interference rings are viewed with a microscope focused on the air film where the rings are formed as shown in Fig. 9.

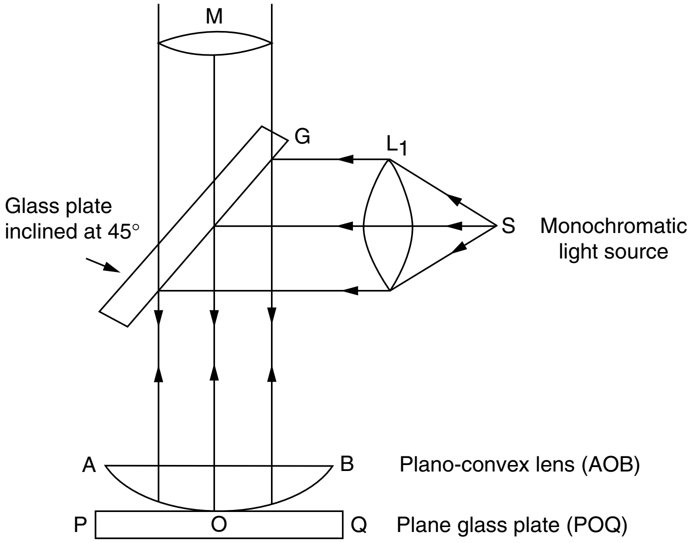


Figure 9 Newton's ring experiment.

3.12.2 Newton's Rings by Reflected Light

The phenomenon of Newton's rings was satisfactorily explained by Young. These rings are produced as a result of interference between the light waves reflected from the upper and the lower surfaces of the air film formed between the plano-convex lens and the plane glass plate as shown in Fig. 10.

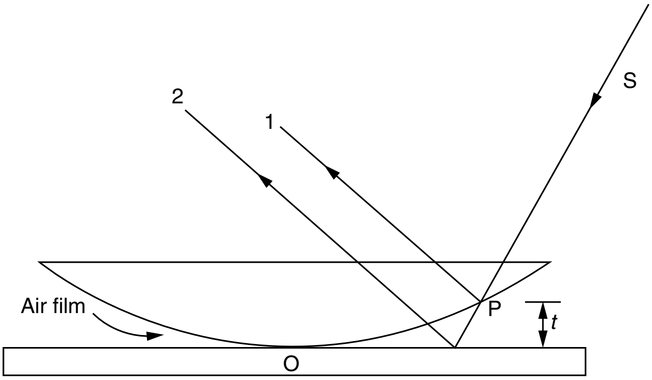


Figure 10 Formation of rings.

The effective path difference between the interfering rays in reflected light is

$$\Delta = 2\mu t\cos(r+\theta) - \frac{\lambda}{2}$$

For normal incidence, $r = 0$ and for a very small angle $\theta = 0$. So

$$\cos(r+\theta) \cong 1$$

Hence, net path difference between rays 1 and 2 is

$$\Delta = 2\mu t - \frac{\lambda}{2}$$

3.12.3 Condition for Constructive Interference or Maximum Intensities or Brightness

If the path difference $\Delta = n\lambda$ where $n = 0, 1, 2, 3, \ldots$, then constructive interference takes place and the film appears bright in the reflected light:

$$2\mu t - \frac{\lambda}{2} = n\lambda \Rightarrow 2\mu t = (2n+1)\frac{\lambda}{2} \tag{3.37}$$

3.12.4 Condition for Destructive Interference or Minimum Intensities or Darkness

If the path difference $\Delta = (2n+1)\lambda/2$ where $n = 0, 1, 2, 3, \ldots$, then destructive interference takes place and the film appears dark in the reflected light:

$$2\mu t - \frac{\lambda}{2} = (2n+1)\frac{\lambda}{2} \Rightarrow 2\mu t = (n+1)\lambda$$

Since n is an integer, therefore $(n + 1)$ can also be taken as n. Thus

$$2\mu t = n\lambda \tag{3.38}$$

3.12.5 Diameters of Dark and Bright Rings

To evaluate the diameter of bright and dark rings, a plano-convex lens is placed on the plane glass plate. Let R be the radius of the curved surface of the lens and t be the thickness of the film at the certain point (see Fig. 11).

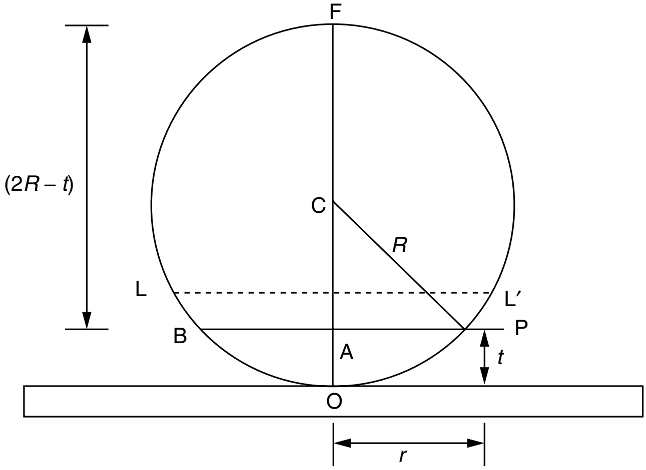


Figure 11 Determination of diameter of Newton's ring.

From the property of the circle

$$\mathrm{AP} \times \mathrm{AB} = \mathrm{OA} \times \mathrm{AF}$$

But AP = AB = r, the radius of the ring passing through P. We have

$$r^2 = t(2R - t) = 2Rt - t^2$$

In actual practice R is quite large and t is very small. Therefore t^2 may be neglected in comparison with $2Rt$. Now

$$r^2 = 2Rt \Rightarrow t = \frac{r^2}{2R} \tag{3.39}$$

For **bright rings**, substituting this value of t from Eq. (3.39) in Eq. (3.37), we get

$$2\mu t = (2n + 1)\,\frac{\lambda}{2}$$

$$\Rightarrow 2\mu\,\frac{r^2}{2R} = (2n + 1)\,\frac{\lambda}{2}$$

$$\Rightarrow r^2 = \frac{(2n+1)\,\lambda R}{2\mu}$$

This denotes the radius of nth bright ring. Thus we have

$$r_n^2 = \frac{(2n+1)\lambda R}{2\mu} \tag{3.40}$$

If D_n is the diameter of the nth bright ring, we have

$$r_n = \frac{D_n}{2}$$

Therefore Eq. (3.40) becomes

$$\left(\frac{D_n}{2}\right)^2 = \frac{(2n+1)\lambda R}{2\mu} \Rightarrow D_n^2 = \frac{4(2n+1)\ \lambda R}{2\mu} \Rightarrow D_n^2 = \frac{2(2n+1)\ \lambda R}{\mu}$$

For air film $\mu = 1$. So

$$D_n^2 = 2(2n+1)\lambda R \Rightarrow D_n = \sqrt{2(2n+1)\ \lambda R} \Rightarrow D_n = \sqrt{2\lambda R}\ \sqrt{2n+1} \Rightarrow D_n \propto \sqrt{2n+1}$$

As $n = 0, 1, 2, 3 \ldots (2n + 1)$ is an odd number, the diameters of successive bright rings are proportional to the square root of the odd natural numbers.

For **dark rings**, substituting the value of t in Eq. (3.38), we get

$$2\mu\frac{r_n^2}{2R} = n\lambda \Rightarrow r_n^2 = \frac{n\lambda R}{\mu}$$

If D_n is the diameter of the nth dark ring, we have

$$r_n = \frac{D_n}{2}$$

Therefore

$$D_n^2 = \frac{4n\lambda R}{\mu}$$

For air film $\mu = 1$. So

$$\begin{aligned} D_n^2 = 4n\lambda R &\Rightarrow D_n = \sqrt{4n\lambda R} \\ &\Rightarrow D_n = \sqrt{4\lambda R}\sqrt{n} \\ &\Rightarrow D_n = \sqrt{4\lambda R} \end{aligned}$$

Thus the diameter of successive dark rings is proportional to the square root of the natural numbers.

3.12.6 Determination of Wavelength of Light Used

We know that the diameter of nth dark ring is given by

$$D_n^2 = 4n\lambda R \tag{3.41}$$

Diameter of $(n + p)$th dark ring is given by

$$D_{n+p}^2 = 4(n+p)\lambda R \tag{3.42}$$

From Eqs. (3.41) and (3.42), we have

$$D_{n+p}^2 - D_n^2 = 4(n+p)\lambda R - 4n\lambda R \Rightarrow D_{n+p}^2 - D_n^2 = 4p\lambda R$$

$$\Rightarrow \lambda = \frac{D_{n+p}^2 - D_n^2}{4pR} \tag{3.43}$$

Equation (3.43) represents the wavelength of light used.

3.13 Determination of the Refractive Index of a Liquid

In order to determine the refractive index of given liquid, first the experiment is performed in air. So

$$(D_{n+p}^2 - D_n^2)_{\text{air}} = 4p\lambda R \tag{3.44}$$

Now the liquid whose refractive index (μ) is to be determined is introduced between the plano-convex lens and plane glass plate. Hence,

$$(D_{n+p}^2 - D_n^2)_{\text{liquid}} = 4p\lambda R/\mu \tag{3.45}$$

From Eqs. (3.44) and (3.45), we have

$$\mu = \frac{(D_{n+p}^2 - D_n^2)_{\text{air}}}{(D_{n+p}^2 - D_n^2)_{\text{liquid}}}$$

This is required expression for refractive index of liquid.

Solved Examples

Example 1

Prove that $\dfrac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \dfrac{2\sqrt{\alpha}}{1+\alpha}$, where α is the ratio of two intensities.

Solution: Let us consider I_1 and I_2 to be the intensities and a and b the amplitudes of the two coherent sources. Then according to the question

$$\frac{I_1}{I_2} = \frac{a^2}{b^2} \quad \Rightarrow \frac{a}{b} = \sqrt{\frac{I_1}{I_2}} = \sqrt{\alpha}$$

Now

$$I_{\max} = (a+b)^2 \quad \text{and} \quad I_{\min} = (a-b)^2$$

So

$$\frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{(a+b)^2 - (a-b)^2}{(a+b)^2 + (a-b)^2} = \frac{2ab}{a^2+b^2} = \frac{2\left(\dfrac{a}{b}\right)}{\left(\dfrac{a}{b}\right)^2 + 1} = \frac{2\sqrt{\alpha}}{1+\alpha}$$

Example 2

In an interference pattern with two coherent sources the amplitude of the intensity variation is found to be 5% of the average intensity. Calculate the relative intensities of the interfering sources.

Solution: Consider the amplitude ratio of the two sources to be a:1. Then I_{max} and I_{min} is $(a+1)^2$ and $(a-1)^2$, respectively,

$$\frac{I_{\text{max}}}{I_{\text{min}}} = \frac{(a+1)^2}{(a-1)^2}$$

Since, intensity variation is found at 5%, then the maximum is 105 and minimum is 95. So

$$\frac{I_{\text{max}}}{I_{\text{min}}} = \frac{(a+1)^2}{(a-1)^2} = \frac{(105)^2}{(95)^2}$$

$$\Rightarrow \frac{a+1}{a-1} = 1.05 \Rightarrow a \cong 40$$

Therefore,

$$\frac{I_1}{I_2} = \frac{a^2}{1} = \frac{(40)^2}{1} = \frac{1600}{1}$$

Example 3

Light of wavelength 5893 Å is reflected at nearly normal incidence from a soap film of refractive index 1.42. What is the least thickness of the film that will appear black?

Solution: We know that the condition for dark ring or black in reflected light is

$$2\mu t \cos r = n\lambda$$

According to the question, $n = 1$ (for least thickness), $\lambda = 5893$ Å, $\mu = 1.42$. For normal incidence $r = 0$. Hence

$$t = \frac{1 \times 5893}{2 \times 1.42 \times 1} = 2075 \text{ Å}$$

Example 4

Two glass plates enclose a wedge-shaped air film, touching at one edge and separated by a wire of 0.05 mm diameter at a distance of 10 cm from the edge. Calculate the fringe width of $\lambda = 5500$ Å from a broad source that falls normally on the film.

Solution: Fringe width

$$\omega = \frac{\lambda}{2\mu\theta_n}$$

We have $\lambda = 5500\ \overset{o}{A} = 5500 \times 10^{-8}$ cm, $t = 0.05$ mm $= 0.005$ cm, distance $(x) = 10$ cm and for air film $\mu = 1$.

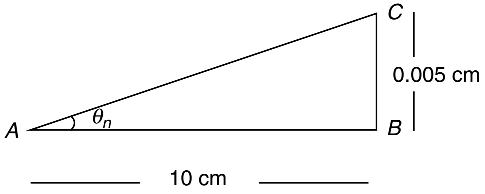


Figure 12 Wedge-shaped film.

From Fig. 12

$$\theta_n = \frac{BC}{AB} = \frac{0.005}{10}$$

So

$$\omega = \frac{\lambda}{2\mu\theta_n} = \frac{5500 \times 10^{-8} \times 10}{2 \times 0.005} = 5.5 \times 10^{-3} \text{ cm}$$

Example 5

If the angle of wedge is 0.15° of arc and the wavelength of sodium D lines are 5890 Å and 5896 Å, find the distance from the apex of the wedge at which the maximum due to each wavelength first coincide.

Solution: If the two wavelengths are coincide and t is the thickness of the film. Thus,

$$2t = (2n + 1)\lambda_1/2 = (2n + 3)\lambda_2/2$$

So,

$$n = \frac{(3\lambda_2 - \lambda_1)}{2(\lambda_1 - \lambda_2)}$$

Now

$$2t = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

Suppose θ_n is the angle of wedge and x is the distance from the apex of the wedge. Then $t = x\,\theta_n$. So

$$2x\,\theta_n = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}$$

Substituting $\lambda_1 = 5896 \times 10^{-8}$ cm, $\lambda_2 = 5896 \times 10^{-8}$ cm and $\theta_n = 0.15° = 0.15\ \pi/180$ radian in above equation, we get

$$x = \frac{5896 \times 10^{-8} \times 5890 \times 10^{-8} \times 180}{(5896 \times 10^{-8} - 5890 \times 10^{-8}) \times 2 \times 0.15 \times 3.14} = 11.05 \text{ cm}$$

Example 6

Newton's rings are made with light of wavelength $\lambda = 6400$ Å and a thin layer of oil with $\mu = 1.60$ formed between the curved surface of a plano-convex lens (radius of curvature 80 cm, $\mu = 1.65$) and a plane glass plate with $\mu = 1.55$. Calculate the radius of the smallest dark ring.

Solution: We have the condition for smallest dark ring

$$2\mu_{\text{oil}}\, t = \frac{\lambda}{2} \tag{3.46}$$

If r is the radius of ring, then

$$t = \frac{r^2}{2R} \quad \text{or} \quad 2\mu_{\text{oil}}\, t = \frac{r^2}{R} \tag{3.47}$$

From Eqs. (3.46) and (3.47), we get

$$r = \sqrt{\frac{\lambda R}{2\mu_{\text{oil}}}}$$

Given $R = 100$ cm, $\lambda = 6000$ Å $= 6400 \times 10^{-8}$ cm and a thin layer of oil $\mu = 1.60$. So

$$r = \sqrt{\frac{100 \times 6000 \times 10^{-8}}{2 \times 1.60}} = 0.0433 \text{ cm}$$

Example 7

Newton's rings are observed in reflected light of wavelength 6000 Å. The diameter of the 10th dark ring is 0.5 cm. Find the radius of curvature of the lens and the thickness of the corresponding air film.

Solution: The diameter of the dark ring is

$$D_n^2 = 4n\lambda R$$

We have $D_n = 0.5$ cm, $n = 10$ and $\lambda = 6000 \times 10^{-8}$ cm. So

$$R = \frac{D_n^2}{4n\lambda} = \frac{(0.5)^2}{4 \times 10 \times 6000 \times 10^{-8}} = 104 \text{ cm}$$

The thickness of the film in terms of diameter is given by

$$t = \frac{D_n^2}{8R} = \frac{(0.5)^2}{4 \times 104} = 3.14 \times 10^{-4} \text{ cm}$$

Short Answers of Some Important Questions

1. What is interference?

   Answer: When two waves of the same frequency and constant phase difference travel simultaneously in the same direction, then there is a change in the intensity of the waves due to superposition of two waves. This change in the intensity is said to be interference.

2. What do you understand by constructive and destructive interference?

   Answer: Some points where the change in intensity is greater than the sum of the intensities due to the individual waves is called constructive interference whereas some other points where change in intensities is less than the sum of the intensities due to individual waves is called destructive interference.

3. What do you understand coherent sources?

   Answer: If the phase difference is constant between the two sources, then they are said to be coherent.

4. Why can two independent sources not be coherent?

   Answer: Two independent sources cannot be coherent because they cannot maintain constant phase difference between them.

5. What do you understand by wedge-shaped film?

   Answer: If two surfaces are inclined at angle θ at one end and other end has regularly varying thickness, then they form wedge-shaped film.

6. What are Newton rings?

   Answer: When a plano-convex surface is placed on a glass plate, an air film of gradually increasing thickness is formed between the two. When monochromatic light is allowed to fall normally on the film and viewed in reflected light, alternate dark and bright rings are observed. These are Newton's rings.

7. Why are Newton rings circular?

   Answer: These rings are foci of constant thickness of the air film and since the foci are concentric circle, hence fringes are circular.

8. What is the importance of extended source?

   Answer: To view the whole thin film, an extended source is necessary.

9. What happens when the order of rings increases?

   Answer: The rings get closer when order of rings increases. This is due to the fact that the radii of dark rings are proportional to square root of natural numbers while those of bright rings are proportional to square root of odd natural numbers.

10. Why is central ring dark?

    Answer: At the point of contact the two interfering rays are opposite in phase and produce zero intensity.

Important Points and Formulas

1. When two waves of same frequency and constant phase difference travel simultaneously in the same direction, then there is a change in the intensity of the waves due to superposition of two waves. This change in the intensity is said to be interference.
2. The resultant amplitude at a point and at any instant of time is the algebraic sum of the amplitudes of the individual waves. This is known as a principle of superposition.
3. Path difference $= \dfrac{\lambda}{2\pi}$ Phase difference.
4. Coherent sources are nothing but two light waves of same frequency of wavelength having same amplitude and always a constant phase difference between them.

5. Fringe width (ω) is given by

$$\omega = x_n - x_{n-1} = \frac{(2n-1)\,D\lambda}{4d} - \frac{[2(n-1)-1]\,D\lambda}{4d}$$
$$= \frac{D\lambda}{2d}$$

6. Fringe width (separation between two consecutive bright or dark fringes) $\omega = \dfrac{\lambda}{2\mu\theta_n}$

7. The diameters of successive bright rings are proportional to the square root of the odd natural numbers.

8. The diameters of successive dark rings are proportional to the square root of the natural numbers.

9. The wavelength of light used in Newton's ring experiment is

$$\lambda = \frac{D_{n+p}^2 - D_n^2}{4pR}$$

10. The refractive index is

$$\mu = \frac{(D_{n+p}^2 - D_n^2)_{\text{air}}}{(D_{n+p}^2 - D_n^2)_{\text{liquid}}}$$

Multiple Choice Questions

1. The phenomenon's of interference of light has proved
   - **(a)** wave nature
   - **(b)** particle nature
   - **(c)** wave and particle and both
   - **(d)** None of these
2. In interference the energy is only transferred from the points of
   - **(a)** maximum to the minimum displacement
   - **(b)** minimum to the maximum displacement
   - **(c)** Both (a) and (b)
   - **(d)** None of these
3. Interference fringes are
   - **(a)** of different widths
   - **(b)** independent of width
   - **(c)** of equal width
   - **(d)** None of these
4. In an interference pattern, the intensity at a point is maximum if the phase difference between the two interfering light beams at that point is
   - **(a)** even multiple of π
   - **(b)** odd multiple of π
   - **(c)** integral multiple of π
   - **(d)** None of these
5. In an interference pattern, the intensity at a point is minimum if the phase difference between the two interfering light beams at that point is
   - **(a)** even multiple of π
   - **(b)** odd multiple of π
   - **(c)** integral multiple of π
   - **(d)** None of these
6. Which of the following device is an example of division of wave front?
   - **(a)** Newton's ring
   - **(b)** Fresnel's mirrors
   - **(c)** Michelson interferometer
   - **(d)** None of these
7. The actual shape of interference fringes in Young's double slit experiment is
   - **(a)** elliptical
   - **(b)** parabolic
   - **(c)** hyperbolic
   - **(d)** circle
8. Two independent sources should not be coherent because they
   - **(a)** cannot have constant phase difference
   - **(b)** cannot have different phase difference
   - **(c)** are independent of phase different
   - **(d)** None of these
9. The wedge fringe pattern will always begin with
   - **(a)** dark fringe
   - **(b)** bright fringe
   - **(c)** maximum intensity
   - **(d)** None of these

Short Answer Type Questions

1. What do you meant by interference?
2. Explain the types of interference.
3. What do you understand by coherent sources?
4. How are coherent sources obtained in practice?
5. Discuss why two independent sources of light of same wavelength cannot show interference.
6. What is thin film interference?
7. Explain the maxima and minima in thin film interference due to reflected light.
8. Why are Newton rings circular?

Long Answer Type Questions

1. What do you understand by interference? Explain types of interference.
2. Explain the interference film in thin film due to reflected light.
3. What is a wedge-shaped film? Determine the condition for maxima and minima.
4. Prove that interference in thin film due to reflected light is complementary with transmitted light.
5. What do you understand by Newton's rings? Explain their experimental arrangement.
6. Explain the formation of Newton's rings.
7. How can you determine the wavelength of light used by Newton's rings?
8. Determine the diameter of rings for maximum and minimum intensity.

Numerical Problems

1. If in an interference pattern, the ratio between maximum and minimum intensities is 36:1, find the ratio between the amplitude and intensities of the two interfering waves.
2. In an interference pattern, the amplitude of intensity variation is found to be 10% of the average intensity. Calculate the relative intensities of the interfering sources.
3. In a Young's double slit experiment the angular width of a fringe termed on a distant screen is 0.1°. The wavelength of light used is 6000 Å. What is the spacing between the slits?
4. Calculate the thickness of the thinnest film ($\mu = 1.5$) in which interference of violet component ($\lambda = 4000$ Å) of incident light can take place by reflection.
5. Two glass plates enclose a wedge-shaped air film, touching at one edge and are separated by a wire of 0.03 mm diameter at a distance of 12 cm from the edge. Calculate the fringe width of $\lambda = 6000$ Å from a broad source that falls normally on the film.
6. If the angle of wedge is 0.25° of arc and the wavelength of sodium D lines are 5890×10^{-8} and 5896×10^{-8} cm, find the distance from the apex of the wedge at which the maximum due to each wavelength first coincide.
7. Newton's rings are made with light $\lambda = 6400$ Å and a thin layer of oil $\mu = 1.60$ formed between the curved surface of a plano-convex lens (radius of curvature 80 cm, $\mu = 1.65$) and a plane glass plate $\mu = 1.55$. Calculate the radius of the smallest dark ring.
8. In an arrangement for observing Newton's rings with two different media between the glass surfaces, the *n*th rings have diameters as 10: 7. Find the ratio of the refractive indices of the two media.

Answers

Multiple Choice Questions

1. (a)
2. (b)
3. (c)
4. (a)
5. (b)
6. (b)
7. (c)
8. (a)
9. (a)

Numerical Problems

1. $I_1:I_2 = 49:25$
2. $I_1:I_2 = 400:1$
3. 3.44×10^{-2} cm
4. 1.33×10^{-5} cm
5. 1.08×10^{-6} cm
6. 6.63 cm
7. 0.04 cm
8. 49 : 100

4

Diffraction of Light

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Single, double and *N*-slit diffraction.
- Diffraction grating.
- Grating spectra.
- Dispersive power.
- Rayleigh's criterion.
- Resolving power of grating.

4.1 Introduction

We have seen that the sunlight comes in a dark room through a hole in the window in straight line. Similarly, a sharp shadow of an opaque object implies rectilinear propagation of light. But in 1665, Grimaldi observed that when a beam of light passes through a small aperture or a narrow slit, it does not follow rectilinear path but bends around the corners of the obstacles (slit or aperture). This bending of light depends on the size of slit or aperture and wavelength of light wave. This bending or deviation is extremely small when the wavelength is small in comparison to size of slit or aperture and much more if wavelength is comparable to size of slit or aperture. Thus the diffraction phenomenon is the bending of light from the edges or corners of slit or obstacle and spreading in the region or geometrical shadow and distribution of intensity in the form of bright and dark fringes on screen which is called diffraction pattern (see Fig. 1).

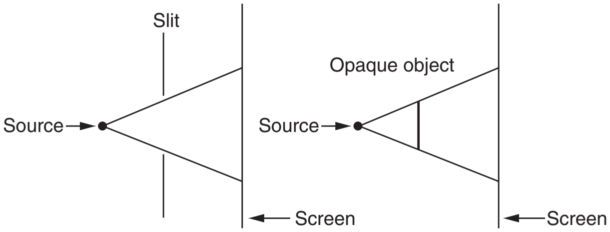


Figure 1 The bending of light round corners of an object.

4.2 Classification of Diffraction

The phenomena of diffraction can be classified into two following categories:

1. **Fresnel diffraction:** In Fresnel diffraction, the source or screen or both are at finite distance from obstacles.
2. **Fraunhofer diffraction:** In Fraunhofer diffraction, the source or screen or both are at infinite distance from obstacles.

The differences between Fresnel and Fraunhofer diffractions are given in Table 1.

Table 1 Differences between Fresnel and Fraunhofer diffractions

S. No.	*Fresnel Diffraction*	*Fraunhofer Diffraction*
1.	Source or screen or both are at finite distance from obstacle.	Source or screen or both are at infinite distance from obstacle.
2.	No lens is used.	Combination of lenses is used.
3.	Incident wave front is generally spherical or cylindrical.	Incident wave front is plane.
4.	Diffraction pattern is a shadow of obstacle.	Diffraction pattern is an image of obstacle.
5.	Central point in diffraction pattern is either dark or bright depending on the number of Fresnel zones.	Central point is always bright.

4.3 An Important Mathematical Analysis

This mathematical treatment is required in the formulation of intensity variation relation in single slit which in turn helps in analysis of double slit and multiple slit (grating) diffraction. Also it gives a glimpse of how mathematical relation having physical reality makes the understanding of a system easy.

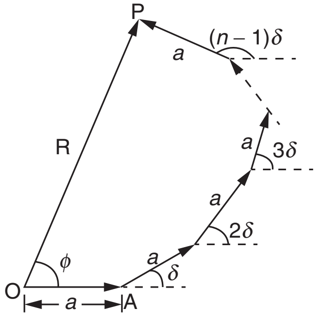


Figure 2 Determination of resultant amplitude and phase.

Here we consider waves having equal amplitude 'a' and having common phase difference δ between successive waves, that is, phase increases in AP. To find the resultant amplitude R and phase θ, a polygon is constructed as shown in Fig. 2. Resolving a along and perpendicular to the first side, we get

$$R\cos\theta = a + a\cos\delta + a\cos 2\delta + a\cos 3\delta + \cdots + a\cos(n-1)\delta \tag{4.1}$$

$$R\sin\theta = 0 + a\sin\delta + a\sin 2\delta + a\sin 3\delta + \cdots + a\sin(n-1)\delta \tag{4.2}$$

Multiplying Eq. (4.1) by 2 sin$\delta/2$ and using trigonometric identities, we get

$$2R\cos\theta\sin\delta/2 = a\,[2\sin(\delta/2) + 2\sin(\delta/2)\cos\delta + 2\sin(\delta/2)\cos 2\delta + \cdots + 2\sin(\delta/2)\cos(n-1)\delta\,]$$

$$= a\,[2\sin(\delta/2) + \{\sin(3\delta/2) - \sin(\delta/2)\} + \{\sin(5\delta/2) - \sin(3\delta/2)\} + \cdots + \{\sin(n-1/2)\delta - \sin(n-3/2)\delta\}]$$

$$= a\,[\sin(\delta/2) + \sin(n-1/2)\delta] = 2a\sin(n\delta/2)\cos(n-1/2)\delta$$

So

$$R\cos\theta = [a\sin(n\delta/2)\cos(n-1)\delta/2]/\sin\delta/2$$

Similarly,

$$R\sin\theta = [a\sin(n\delta/2)\sin(n-1)\delta/2]/\sin\delta/2$$

Squaring and adding the above equations, we have

$$R = a\sin(n\delta/2)/\sin(\delta/2)$$

and $$\tan\theta = \tan(n-1)\delta/2 \Rightarrow \theta = (n-1)\delta/2$$

Let $n\delta = 2\alpha$. Then $R = a\sin\alpha/\sin(\alpha/n)$. α/n is very small as n is infinitely large. Hence

$$R = a\sin\alpha/(\alpha/n) = na\sin\alpha/\alpha = A\sin\alpha/\alpha$$

and $$\theta = (n-1)\delta/2 \approx n\,\delta/2 = \alpha$$

As n is very large then $n \approx n-1$. Since Fraunhofer diffraction is much more important than Fresnel diffraction, therefore in the next section we will discuss the simple diffraction phenomena.

4.4 Fraunhofer Diffraction at a Single Slit

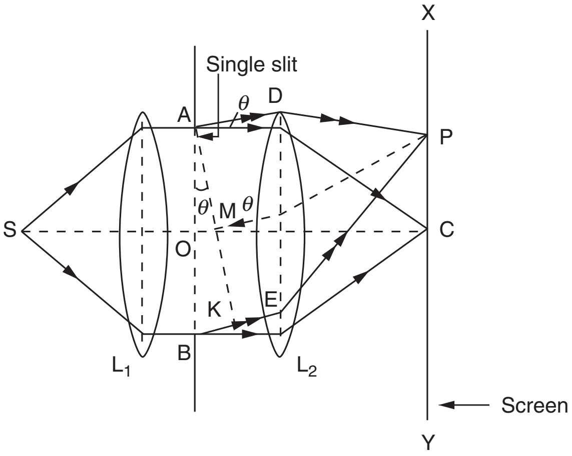


Figure 3 Fraunhofer diffraction at a single slit.

Let a monochromatic light of wavelength λ be incident on collimating lens L_1 (see Fig. 3). A parallel beam of light emerges out from L_1 and normally falls upon a slit AB, whose length is large compared to its width a. The diffracted light is focused by another lens L_2. The diffraction pattern obtained on the screen consists of a central bright band having alternate dark and bright bands of decreasing intensity on both sides.

The rays diffracted along the direction of incident rays are focused at C while those diffracted at an angle θ are focused at P. Since all the wavelets from AB reach C in the same phase, hence intensity at C is maximum whereas the wavelets reach P at different times due to unequal distance. Hence, they have different path and phase.

Let AK be perpendicular to BK. The path difference between the rays originating from extreme points A and B is given by

$$BK = AB\sin\theta = a\sin\theta \tag{4.3}$$

where *a* is the width of slit AB. Now, the corresponding phase difference between the rays originating from extreme points A and B is

$$\frac{2\pi}{\lambda}\times a\sin\theta$$

Let the aperture AB be divided into a large number *n* of equal parts, each part being the source of secondary wavelets. The amplitude of vibrations at P due to each part will be the same, say *a*, but their phase will vary gradually from 0 to $(2\pi/\lambda)\times a\sin\theta$. The phase difference between the waves from two consecutive parts is

$$\delta=\frac{1}{n}\frac{2\pi}{\lambda}\times a\sin\theta \tag{4.4}$$

where *n* is the number of vibrations. The resultant amplitude at P is given by

$$R=a\frac{\sin(n\delta/2)}{\sin(\delta/2)}=a\frac{\sin(\pi a\sin\theta/\lambda)}{\sin(\pi a\sin\theta/n\lambda)} \tag{4.5}$$

$$\Rightarrow \qquad R=a\frac{\sin\alpha}{\sin(\alpha/n)} \tag{4.6}$$

where $\alpha=(\pi a\sin\theta)/\lambda$. Since α/n is very small, therefore, $\sin(\alpha/n)=(\alpha/n)$. So we have

$$R=a\frac{\sin\alpha}{\alpha/n}=na\frac{\sin\alpha}{\alpha}\quad\Rightarrow\quad R=A\frac{\sin\alpha}{\alpha} \tag{4.7}$$

where $A=na$ is the amplitude of all the vibrations in same phase. Now intensity at P is

$$I=R^2=A^2\frac{\sin^2\alpha}{\alpha^2} \tag{4.8}$$

For intensity to be maximum or minimum

$$\frac{dI}{d\alpha}=0\Rightarrow\frac{d}{d\alpha}\left(A^2\frac{\sin^2\alpha}{\alpha^2}\right)=0\Rightarrow A^2\left(\frac{\alpha^2 2\sin\alpha\cos\alpha-\sin^2\alpha\ 2\alpha}{\alpha^4}\right)=0$$

$$\Rightarrow A^2\left(\frac{2\sin\alpha\cos\alpha}{\alpha^2}-\frac{2\sin^2\alpha}{\alpha^3}\right)=0\Rightarrow A^2\frac{2\ \sin\alpha}{\alpha}\left(\frac{\alpha\cos\alpha-\sin\alpha}{\alpha^2}\right)=0$$

Now,

$$\text{Either }\frac{\sin\alpha}{\alpha}=0\quad\text{or}\quad\left(\frac{\alpha\cos\alpha-\sin\alpha}{\alpha^2}\right)=0$$

$$\Rightarrow\text{Either }\sin\alpha=0\quad\text{or}\quad\alpha\cos\alpha-\sin\alpha=0$$

$$\Rightarrow\ \text{(a) }\sin\alpha=0\quad\text{or}\quad\text{(b) }\alpha=\tan\alpha$$

Condition for Minimum Intensity: When $\sin\alpha/\alpha=0$, then intensity is zero. So

$$\sin\alpha=0\Rightarrow\alpha=\pm m\pi\Rightarrow\frac{\pi a\sin\theta}{\lambda}=\pm m\pi$$

$$\Rightarrow \qquad a\sin\theta=\pm m\lambda \tag{4.9}$$

Equation (4.9) gives the position of 1st, 2nd, 3rd,… minima corresponding to $m = 1, 2, 3$ and so on. Here $m \neq 0$ because when $m = 0$ then $a\sin\theta = 0$ which is the condition of maximum intensity.

Condition for Maximum Intensity: For maximum intensity $\alpha = \tan\alpha$. This equation can be solved graphically by plotting the curves $y = \alpha$ and $y = \tan\alpha$ as shown in Fig. 4.

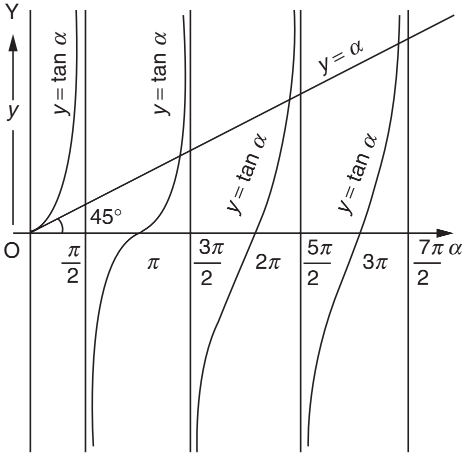


Figure 4 Curves between $y = \alpha$ and $y = \tan\alpha$.

The abscissa of the points of intersection of these curves gives the required value of α for which intensity is maximum.

Principal maximum For central maximum $\alpha = 0$ so $(\sin\alpha)/\alpha \to 1$. From Eq. (4.8) we have

$$I = A^2 = I_0 \text{ (say)}$$

Secondary maxima Secondary maxima falls at the points of intersection of two curves which are nearly at α = odd number multiples of $\lambda/2$:

$$\alpha = \pm\frac{(2m+1)\lambda}{2} \tag{4.10}$$

$$\Rightarrow \frac{\pi a \sin\theta}{\lambda} = \pm\frac{(2m+1)\pi}{2} \Rightarrow a\sin\theta = \pm\frac{(2m+1)\lambda}{2} \tag{4.11}$$

If we put $m = 1, 2, 3, \ldots$ we get the position of 1st, 2nd, 3rd, … secondary maxima, respectively. Now putting $m = 1, 2, 3, \ldots$ in Eq. (4.11) we get

$$\alpha = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

Intensity of 1st secondary maxima

$$I_1 = A^2\frac{\sin^2(3\pi/2)}{(3\pi/2)^2} = A^2\frac{4}{9\pi^2} = \frac{4}{9\pi^2}I_0 \text{ i.e. } 4.5\% \text{ of } I_0$$

Intensity of 2nd secondary maxima

$$I_2 = A^2\frac{\sin^2(5\pi/2)}{(5\pi/2)^2} = A^2\frac{4}{25\pi^2} = \frac{4}{25\pi^2}I_0 \text{ i.e. } 1.5\% \text{ of } I_0$$

Intensity of 3rd secondary maxima

$$I_3 = A^2 \frac{\sin^2(7\pi/2)}{(7\pi/2)^2} = A^2 \frac{4}{49\pi^2} = \frac{4}{49\pi^2} I_0$$

Thus the relative intensities of successive maxima are

$$I_0 : I_1 : I_2 : I_3 \ \ldots = 1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2} \ldots$$

Thus, most of the light is concentrated in the central maxima and intensity of secondary maxima goes on decreasing. In short, we can say that the diffraction pattern consists of a bright central maxima surrounded alternately by minima of zero intensity and feeble secondary maxima of rapidly decreasing intensities.

The intensity distribution curve is shown in Fig. 5.

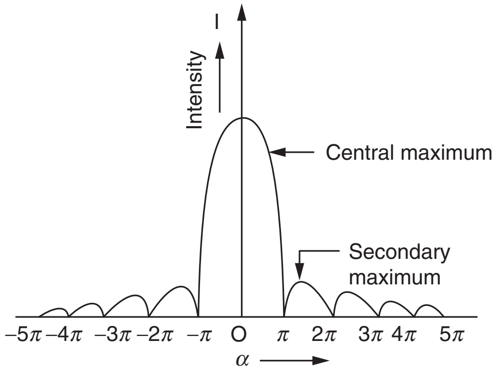


Figure 5 The intensity distribution curve.

4.5 Fraunhofer Diffraction due to Double Slit

Let AB and GH be the two parallel slits of equal width *a* and separated by an opaque distance *b*. Let a plane wave front be incident normally upon the slits. The light diffracted from these slits is focused by lens L_2 on the screen XY as shown in Fig. 6.

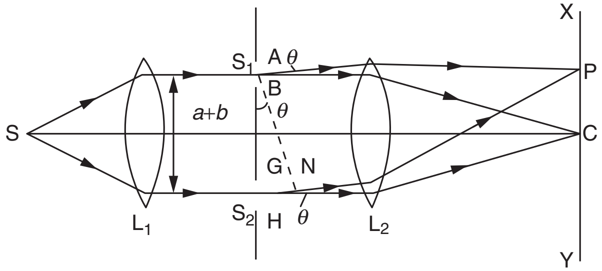


Figure 6 Fraunhofer diffraction due to double slit.

The diffraction at double slit is a case of diffraction as well as interference. The pattern obtained on the screen consists of equally spaced interference fringes in the region normally occupied by the central maxima in the single slit diffraction, while in the region of secondary maxima of single slit diffraction, the diffraction fringes of low intensity are observed.

Basically, if a plane wave front is incident normally on the slits, all the points within the slits act as a source of secondary wave. The secondary waves travelling in the direction of incident light come to focus at C while the other secondary waves travelling in a direction θ with the direction of incident light come to focus at P.

According to theory of diffraction at a single slit, the resultant amplitude R due to all the secondary waves diffracted from each slit along the direction θ is

$$R = A\frac{\sin\alpha}{\alpha}$$

where A is the resultant amplitude due to each single slit in the direction $\theta = 0°$ and $\alpha = (\pi a \sin\theta)/\lambda$. The phase difference between the secondary waves starting from extreme points of a slit is

$$\frac{2\pi a\sin\theta}{\lambda} = 2\alpha$$

In Δ AGN

$$\sin\theta = \frac{\mathrm{GN}}{\mathrm{AG}} = \frac{\mathrm{GN}}{a+b}$$

or Path difference GN = $(a + \mathrm{b})\sin\theta$

and Phase difference $\Phi = \dfrac{2\pi}{\lambda}(a + \mathrm{b})\sin\theta$

The resultant amplitude of both slits (by vector sum; see Fig. 7) is

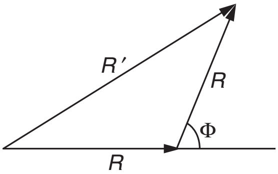


Figure 7 Determination of resultant amplitude.

$$R'^2 = R^2 + R^2 + 2RR\cos\Phi = 2R^2(1+\cos\Phi) = 2R^2\left(2\cos^2\frac{\Phi}{2}\right) = 4R^2\cos^2\frac{\Phi}{2}$$

Putting the value of R and Φ in above equation, we get

$$R'^2 = 4A^2\frac{\sin^2\alpha}{\alpha^2}\cos^2\beta \tag{4.12}$$

where $\beta = \dfrac{\Phi}{2} = \dfrac{\pi}{\lambda}(a+b)\sin\theta$.

The intensity at point 'P' is given by

$$I = 4A^2\frac{\sin^2\alpha}{\alpha^2}\cos^2\beta \tag{4.13}$$

Thus, the resultant intensity at any point depends on two variable factors which are reflected through Fig. 8:

1. The factor $A^2(\sin^2 \alpha)/\alpha^2$ which represents the intensity distribution in diffraction pattern due to any individual slit – diffraction term.
2. The factor $\cos^2 \beta$ which gives the interference pattern due to waves starting from two parallel slits – interference term.

Thus, the resultant intensity at any point is given by the product of these two factors and will be zero if either of these factors is zero.

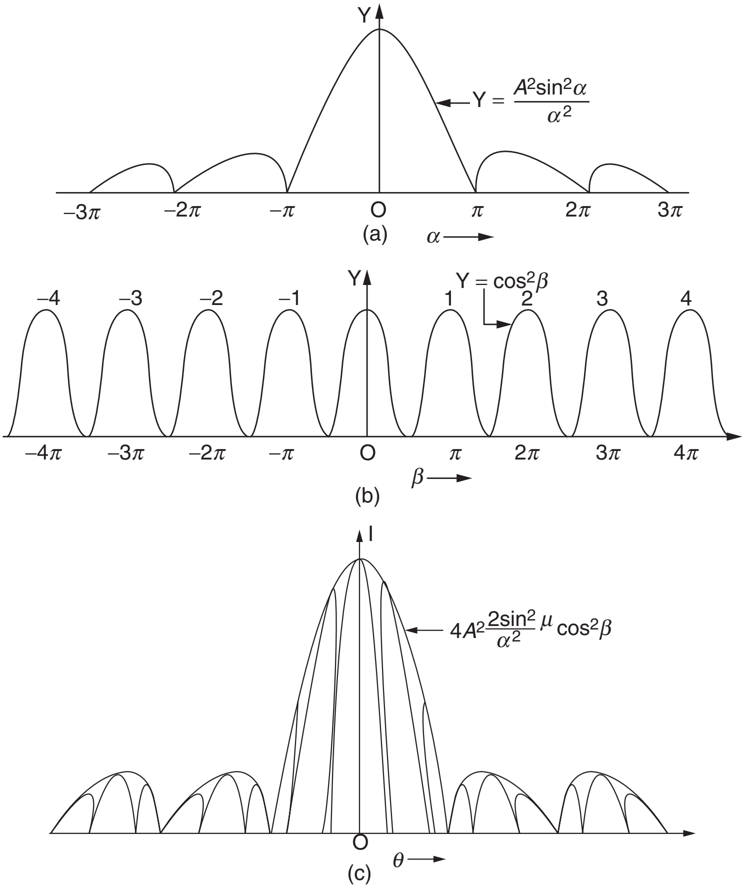


Figure 8 Intensities distribution.

4.5.1 Direction of Maxima and Minima

We can calculate the direction of maxima and minima due to these two terms separately:

1. **Maxima and minima due to diffraction term $A^2(\sin^2 \alpha)/\alpha^2$:** Central maxima at $\alpha = 0$ having alternate minima and secondary maxima.

$$\text{Minima at } \alpha = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

$$\text{Secondary maxima is given by } \alpha = \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

2. **Maxima and minima due to interference term $\cos^2\beta$:**

For maxima $\cos^2\beta = 1$, that is $\beta = \pm n\pi$. So

$$(a+b)\sin\theta = \pm n\lambda$$

Now when $n = 0$, $\theta = 0$ that is we have central maxima. Again

For minima $\cos^2\beta = 0$, that is, $\beta = \pm(2n-1)\pi/2$, so

$$(a+b)\sin\theta = \pm\frac{(2n-1)}{2}\lambda$$

Now, if a is kept constant and b (hence $a+b$) is varied, position of maxima and minima due to diffraction remains unchanged while those due to interference undergo change.

4.6 Condition for Absent Spectra or Missing Spectra

It has been observed that certain order of interference maxima may not appear in the diffraction pattern. These are called absent spectra. The direction of principal maxima is given by

$$(a+b)\sin\theta_n = \pm n\lambda \quad \text{(Interference maxima)} \tag{4.14}$$

where n is order of principal maxima. The direction of diffraction minima is given by

$$a\sin\theta = \pm m\lambda \quad \text{(Diffraction minima)} \tag{4.15}$$

If the values of a and b are such that both Eqs. (4.14) and (4.15) are satisfied simultaneously for same values of θ, then the position of certain interference maxima corresponds to diffraction minima are absent. Dividing Eq. (4.14) by Eq. (4.15) we get

$$\frac{a+b}{a} = \frac{n}{m} \Rightarrow n = \left(\frac{a+b}{a}\right)m \tag{4.16}$$

Equation (4.16) represents the condition for nth order spectrum to be absent from the spectra.

Case 1: When $b = a$, then $n = 2m$. If $m = 1, 2, 3, \ldots$ then $n = 2, 4, 6, \ldots$. Thus the 2nd, 4th, 6^{th}, … order will be absent.

Case 2: When $b = 2a$, then $n = 3m$. If $m = 1, 2, 3, \ldots$ then $n = 3, 6, 9, \ldots$. Thus the 3rd, 6th, 9th, … order will be absent.

4.7 Fraunhofer Diffraction due to *N* Slits or Plane Diffraction Grating

An arrangement consisting of large number of parallel slits of equal width and separated from each other by equal opaque spaces is called a 'diffraction grating'.

A grating is made by ruling a large number of fine equidistant and parallel lines with a diamond point on an optically plane glass plate – true grating. The ruled widths are opaque to the light, while the spaces between any two lines are transparent. Such a grating is called 'transmission grating'. There are about 15,000 lines per inch in such a grating. In practice, the replicas of original gratings – workable gratings – are produced by pouring collodian solution on a true grating which is allowed to harden, then removing the film and fixing it between the glass plates.

Let a wave front of monochromatic light is incident on '*N*' parallel slits each of width '*a*' and separated by opaque space '*b*'.

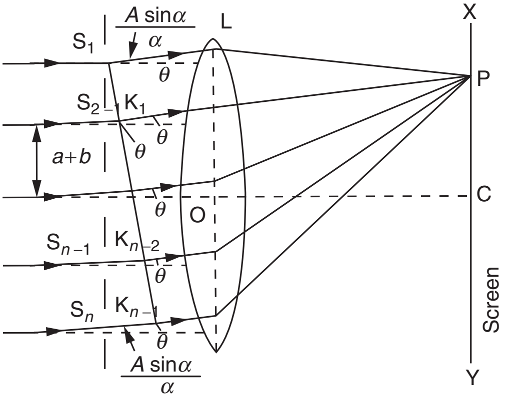


Figure 9 *N*-slits diffraction.

The light diffracted through *N* slits is focused by lens L on the screen XY placed in the focal plane of lens L (see Fig. 9). When the wave front reaches the plane of the slits, each point in the slit sends out secondary wavelets in all directions. From Fraunhofer diffraction at single slit all the secondary waves proceeding from slits in a direction θ are equivalent to a single wave of amplitude

$$R = \frac{A\sin\alpha}{\alpha} \text{ where } \alpha = \frac{\pi a\sin\theta}{\lambda}$$

$$\text{Path difference between two successive waves} = (a+b)\sin\theta$$

$$\text{The corresponding phase difference} = \frac{2\pi}{\lambda}(a+b)\sin\theta = 2\beta$$

As we pass from one vibration to another, the phase goes on increasing by the same amount $(2\pi/\lambda)(a+b)\sin\theta$. Thus, in order to find the amplitude in a direction θ we have to find the resultant amplitude of *N* waves each having amplitude *R* and common phase difference $(2\pi/\lambda)(a+b)\sin\theta$.

Using the standard result, the resultant amplitude in a direction θ is given by

$$R' = R\frac{\sin N\beta}{\sin\beta} = A\frac{\sin\alpha}{\alpha}\frac{\sin N\beta}{\sin\beta}$$

The resultant intensity at point *P* is given by

$$I = R'^2 = \frac{A^2\sin^2\alpha}{\alpha^2}\frac{\sin^2 N\beta}{\sin^2\beta} \tag{4.17}$$

The factor $(A^2\sin^2\alpha)/\alpha^2$ gives the intensity distribution in the diffraction pattern due to a single slit while the factor $\sin^2 N\beta/\sin^2\beta$ gives the distribution of intensity due to interference caused by waves from all the slits.

4.7.1 Direction of Principal Maxima and Minima

The direction of principal maxima is as follows:

$$\sin\beta = 0 \Rightarrow \beta = \pm n\pi$$

$$\Rightarrow \frac{\pi}{\lambda}(a+b)\sin\theta = \pm n\pi \Rightarrow (a+b)\sin\theta = n\lambda \tag{4.18}$$

When $n = 0$ we have $\theta = 0$. This gives the direction of 0th order principal maxima or central maxima. If we put $n = 1, 2, 3, \ldots$ we get 1st, 2nd, 3rd, … order maxima, respectively. The resultant intensity of principal maxima is given by

$$I = A^2 \frac{\sin^2\alpha}{\alpha^2} N^2 \tag{4.19}$$

where

$$\lim_{\beta\to\pm n\pi} \frac{\sin N\beta}{\sin\beta} = N$$

Hence if we increase the number of slits, the intensity of principal maxima increases.

The direction of minima is given by

$$\sin N\beta = 0 \Rightarrow N\beta = \pm m\pi$$

$$\Rightarrow N\frac{\pi}{\lambda}(a+b)\sin\theta = \pm m\pi \Rightarrow N(a+b)\sin\theta = \pm m\lambda \tag{4.20}$$

Here m can take all integral values except $m = 0, N, 2N, 3N, \ldots$ because these values give $\sin\beta = 0$ which gives the position of principal maxima. The ± sign indicates that the minima of a given order lie symmetrically on both sides of the central maxima. It is also clear from Eq. (4.20) that $m = 0$ gives principal maxima of 0th order. $m = 1, 2, 3, \ldots, (N - 1)$ gives minima and $m = N$ gives principal maxima of 1st order.

4.7.2 Direction of Secondary Maxima

The direction of secondary maxima is given by

$$\frac{dI}{d\beta} = 0$$

$$\Rightarrow A^2 \frac{\sin^2\alpha}{\alpha^2} 2\left(\frac{\sin N\beta}{\sin\beta}\right)\frac{N\cos N\beta\sin\beta - \sin N\beta\cos\beta}{\sin^2\beta} = 0$$

$$\Rightarrow N\cos N\beta\sin\beta = \sin N\beta\cos\beta$$

$$\Rightarrow N\tan\beta = \tan N\beta \Rightarrow \tan N\beta = N\tan\beta \tag{4.21}$$

In order to find the intensity of secondary maxima, let us consider Fig. 10 from which we have

$$\sin N\beta = \frac{N\tan\beta}{\sqrt{1+N^2\tan^2\beta}}$$

$$\Rightarrow \quad \frac{\sin^2 N\beta}{\sin^2\beta} = \frac{N^2\tan^2\beta/(1+N^2\tan^2\beta)}{\sin^2\beta}$$

$$= \frac{N^2}{\cos^2\beta(1+N^2\tan^2\beta)} = \frac{N^2}{\cos^2\beta + N^2\sin^2\beta}$$

$$= \frac{N^2}{1-\sin^2\beta+N^2\sin^2\beta} = \frac{N^2}{1+(N^2-1)\sin^2\beta}$$

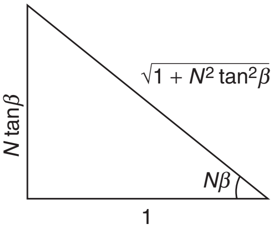


Figure 10 Determination of $N\beta$.

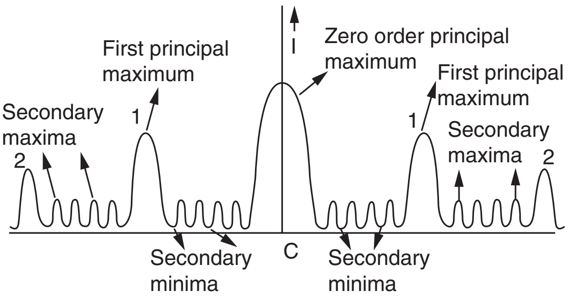


Figure 11 The intensity distribution curve.

Thus the intensity of secondary maxima is given by

$$I = A^2 \frac{\sin^2 \alpha}{\alpha^2} \frac{N^2}{1+(N^2-1)\sin^2 \beta} \tag{4.22}$$

From Eqs. (4.19) and (4.22)

$$\frac{\text{Intensity of secondary maxima}}{\text{Intensity of principal maxima}} = \frac{1}{1+(N^2-1)\ \sin^2\beta} \tag{4.23}$$

Hence as N increases, the intensity of secondary maxima decreases. When N is very large, as in case of diffraction grating, the secondary maxima are not visible in the spectrum. In such cases there is uniform darkness between any two consecutive principal maxima (see Fig. 11).

4.7.3 Width of Principal Maxima

The angular width of principal maxima of any order is defined as the angular separation between the first two minima lying on its either side. The direction of nth order principal maxima is given by

$$(a+b)\sin\theta_n = \pm\, n\lambda \tag{4.24}$$

If $(\theta_n + d\theta_n)$ and $(\theta_n - d\theta_n)$ represent the direction of 1st outer and inner sided minima adjacent to the nth maxima, then $d\theta_n$ will be the angular half width of nth principal maxima.

The direction of minima are given as

$$N(a+b)\sin\theta = m\lambda \tag{4.25}$$

As the first outer and inner sided minima are adjacent to nth order principal maxima, these are obtained in the direction $\theta_n \pm d\theta_n$. This corresponds to $m = nN \pm 1$. Hence from above equation, we have

$$N(a+b)\sin(\theta_n \pm d\theta_n) = (nN \pm 1)\lambda \tag{4.26}$$

For small values of $d\theta_n$, $\cos d\theta_n = 1$ and $\sin d\theta_n = d\theta_n$. So the above equation becomes

$$N(a+b)\sin\theta_n \pm N(a+b)\cos\theta_n d\theta_n = nN\lambda \pm \lambda \tag{4.27}$$

From Eq. (4.25) we have

$$N(a+b)\sin\theta_n = Nn\lambda \tag{4.28}$$

From Eqs. (4.27) and (4.28)

$$nN\lambda \pm\ N(a+b)\cos\theta_n d\theta_n = nN\lambda \pm \lambda$$

$$\Rightarrow d\theta_n = \frac{\lambda}{N(a+b)\cos\theta_n}$$

Thus, the angular half width is inversely proportional to N. If N is sufficiently large, the angular half width will be small and hence the principal maxima will be sharp.

The width of nth order principal maxima is

$$2\, d\theta_n = \frac{2\lambda}{N(a+b)\cos\theta_n}$$

4.7.4 Formation of Spectrum with Grating

The direction of principal maxima is

$$(a+b)\sin\theta_n = \pm n\lambda$$

where a is the width of transparent portion, b is the width of opaque portion and $(a+b)$ is the grating element. From the above equation, we have the following:

1. For a particular value of λ, the directions of principal maxima of different orders are different.
2. For a given value of n, the angle of diffraction θ_n varies with the wavelength. The angle of diffraction increases with increase in λ. As $\lambda_R > \lambda_v$, therefore $\theta_{n(R)} > \theta_{n(v)}$. Hence if white light is incident normally on a grating, each order will contain principal maxima of different wavelengths in different directions as shown in Fig. 12.

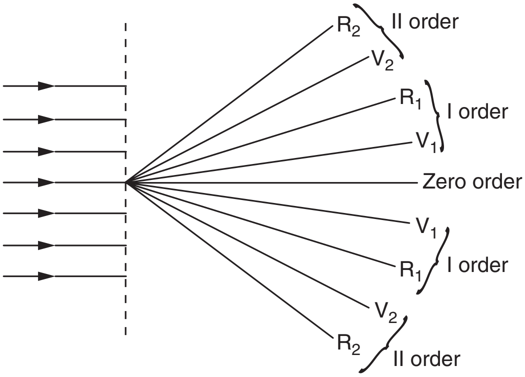


Figure 12 Grating spectra.

It is also clear that for $n = 0$ we have $\theta = 0$ for all values of λ, that is, the 0th order principal maxima for all wavelengths lie in the same direction. Thus the 0th order principal maxima will be white. The 1st order principal maxima of all the wavelengths form the 1st order spectrum. Similarly, the 2nd order principal maxima of all the wavelengths form the 2nd order spectrum and so on. Most of the light is concentrated in the principal maxima of 0th order.

Thus, the spectrum consists of white maxima of 0th order having on either side of it the 1st order spectra, the 2nd order spectra and so on. The spectra of each order consist of colors from violet to red.

4.7.5 Condition for Absent Spectra or Missing Spectra in a Grating Spectrum

The direction of principal maxima obtained with grating is

$$(a+b)\sin\theta_n = \pm n\lambda \tag{4.29}$$

where n is the order of principal maxima. The direction of diffraction minima is

$$a\sin\theta = \pm m\lambda \tag{4.30}$$

If the values of a and b are such that both Eqs. (4.29) and (4.30) are satisfied simultaneously for same values of θ, then the position of certain interference maxima corresponds to diffraction minima are absent.

Dividing Eq. (4.29) by Eq. (4.30), we have

$$\frac{a+b}{a} = \frac{n}{m} \Rightarrow n = \left(\frac{a+b}{a}\right)m \tag{4.31}$$

Equation (4.31) represents the condition for nth order spectrum to be absent from grating spectra.

Case 1: When $b = a$, then $n = 2m$. If $m = 1, 2, 3, \ldots$ then $n = 2, 4, 6, \ldots$. Thus, the 2nd, 4th, 6th, … orders will be absent.

Case 2: When $b = 2a$, then $n = 3m$. If $m = 1, 2, 3, \ldots$ then $n = 3, 6, 9, \ldots$. Thus the 3rd, 6th, 9th, … orders will be absent.

4.8 Dispersive Power of Diffraction Grating

The dispersive power of a diffraction grating is defined as the rate of change of angle of diffraction with the wavelength of light. If $d\theta$ is change in the angle of diffraction corresponding to the light of wavelength $d\lambda$, then $d\theta/d\lambda$ is known as dispersive power.

For a plane transmission grating

$$(a+b)\sin\theta_n = \pm n\lambda \tag{4.32}$$

Differentiating Eq. (4.32) w.r.t. λ, we get

$$(a+b)\cos\theta\frac{d\theta}{d\lambda} = n \tag{4.33}$$

Therefore, dispersive power

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

$$\Rightarrow \frac{d\theta}{d\lambda} = \frac{n}{(a+b)\sqrt{1-\sin^2\theta}} = \frac{n}{(a+b)\sqrt{1-\left(\frac{n\lambda}{(a+b)}\right)^2}}$$

$$\Rightarrow \frac{d\theta}{d\lambda} = \frac{1}{\sqrt{\left(\frac{a+b}{n}\right)^2 - \lambda^2}}$$

where $d\theta$ is the angular separation between two lines having wavelength difference $d\lambda$. From the above equation we can conclude the following:

1. The dispersive power is directly proportional to the order of the spectrum, n. Thus higher the spectrum, greater is the dispersive power. For this reason, the angular separation of two spectral lines is double in 2nd order spectrum in comparison to 1st order spectrum.

2. The dispersive power is inversely proportional to grating element $(a+b)$. This means that it is directly proportional to the number of lines/cm. Thus the angular dispersive power of two given lines is greater with a grating having larger number of lines/cm.
3. The dispersive power is inversely proportional to cos θ. Thus larger the value of θ greater is the dispersive power.

4.8.1 Determination of Grating Element (a + b)

On a grating the number of rulings/inch is given by the manufacturer. If N is the number of rulings/inch then

$$N(a+b) = 1'' = 2.54 \text{ cm} \quad \Rightarrow \quad (a+b) = 2.54/N \text{ cm}$$

4.9 Difference Between Prism and Grating Spectra

The differences between prism and grating spectra are as follows:

1. The prism spectrum is produced by dispersion while grating spectrum is produced by diffraction.
2. A prism gives only one spectrum but a grating gives a number of spectra of different order on both sides of central maxima.
3. The prism spectrum is brighter than the grating spectrum.
4. In case of prism, the deviation is least for red and greatest for violet color while in case of grating, the deviation is least for violet and greatest for red. Thus, the order of colors in the two spectra is opposite.
5. The dispersive power of a grating is

$$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$

which is constant for a given order. Thus the spectral lines are evenly distributed. Hence, the spectrum obtained with a grating is said to be *rational*.

The dispersive power of a prism is $d\mu/(\mu-1)$ which has a higher value in the violet region than in the red region of spectrum. Hence, spreading of lines increases as we move from red end to violet end, that is, spectral lines are not evenly distributed. Hence, the spectrum obtained with a prism is said to be *irrational*.

6. The resolving power of diffraction grating is much greater than that of a prism. Hence the same two nearby spectral lines appear better resolved with a grating than with a prism.
7. The prism spectrum depends upon the material of the prism while grating spectrum is independent of the material.

4.10 Resolving Power

When two objects are very close to each other or very far from our eye, then our naked eye may not be able to see those objects separately and an optical instrument is required to see them as separate. The ability of an optical instrument to just resolve the images of two nearby point sources/objects is called its resolving power. But in case of instruments like prism or grating, it is the ability to just resolve two close spectral lines.

An optical instrument is said to have resolved the two point sources/spectral lines when their corresponding diffraction pattern are sufficiently separated from each other so that the two images can be distinguished from each other.

4.11 Rayleigh's Criterion for Resolution

According to Lord Rayleigh, the two point sources or two spectral lines of equal intensity are just resolved by an optical instrument when the central maxima of diffraction pattern due to one falls on the first minimum of the diffraction pattern of the other. The two curves then intersect each other, just at the midpoint (as intensity is equal) between the two principal maxima.

In Fig. 13(c), A and B are the central maxima of the diffraction patterns of two spectral lines of wavelengths λ and $\lambda + d\lambda$. The difference in the angle of diffraction is large and the two images can be seen as separate ones. Hence, the two spectral lines will appear well resolved.

In Fig. 13(b), the position of central maxima of A coincides with the position of first minima of B. Similarly, the position of central maxima of B coincides with the position of first minima of A. Further, the resultant intensity curve shows a dip at C, that is, in the middle of A and B. The intensity at C is approximately 20% less than that at A and B. The spectral lines can be distinguished from one another and are said to be just resolved.

In Fig. 13(a), the central maxima corresponding to wavelengths λ and $\lambda + d\lambda$ are very close. As the angle of diffraction for first minima of A is greater than the angle of diffraction for central maxima of B, the two lines overlap and they cannot be distinguished as separate images. In this case, the two spectral lines are not resolved.

All types of optical instruments have single aperture; hence expression for the intensity variation due to diffraction with single slit is $I = A^2\sin^2\alpha/\alpha^2$ with $\alpha = \pm\pi$ for the first minima on either side of central maxima but here we take $\alpha = \pi$ on one side.

Further, at the point of intersection that lies at the midpoint we have $\alpha = \pi/2$. Hence,

$$I_1 = I_2 = A^2 \sin^2(\pi/2)/(\pi/2)^2 = 4A^2/\pi^2$$

Thus, the resultant intensity at the midpoint is due to the contribution of intensity of each object. Also we know that intensity adds like scalars. Hence,

$$I_{\text{mid}} = I_1 + I_2 = 8A^2/\pi^2$$

But $I_{\max} = A^2 = I_0$. Hence, finally $I_{\text{mid}}/I_{\max} = 8/\pi^2 = 0.81$, that is, the two spectral lines/two images are said to be just resolved when the intensity at the dip in the middle is 81% of the intensity at either maxima.

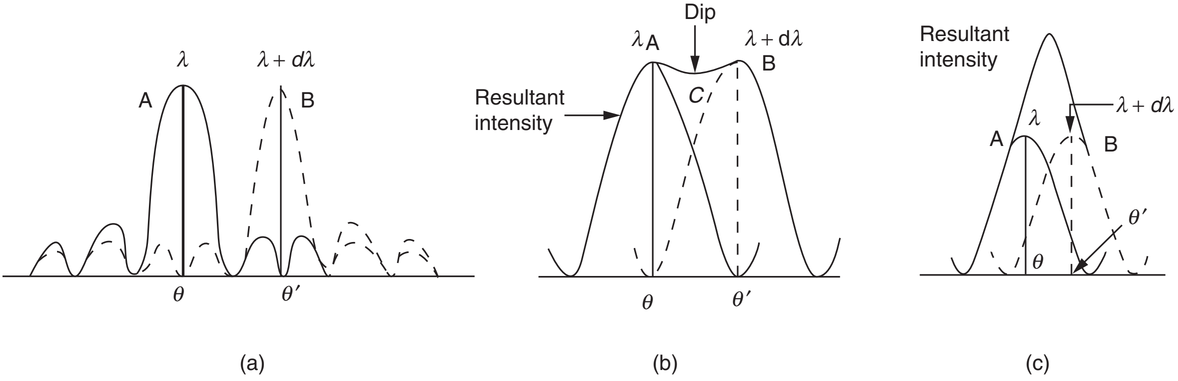


Figure 13 The intensity distribution curves of two wavelengths λ and $\lambda + d\lambda$.

Notes:

1. If resultant intensity is more than 81% of either maxima, then it is not resolved
2. If resultant intensity is equal to 81% of either maxima, then it is just resolved
3. If resultant intensity is less than 81% of either maxima, then it is fully resolved

4.12 Resolving Power of Plane Transmission Grating

Resolving power of grating can be defined as the ratio of the wavelength of either spectral line with the smallest wavelength difference of two close spectral lines which can be just resolved. Let a beam have wavelengths λ and $\lambda + d\lambda$ (as in sodium light we have two wavelengths 5890 Å and 5896 Å ; here $\lambda = 5890$ Å with $d\lambda = 6$ Å; see Fig. 14). We have to form the expression for the resolving power of the diffraction grating, that is to say finding the limiting value of $d\lambda$ out of λ. For this purpose, we put the Rayleigh's criterion and for that matter have to discuss positions of principal maxima and first minima adjacent to principal maxima.

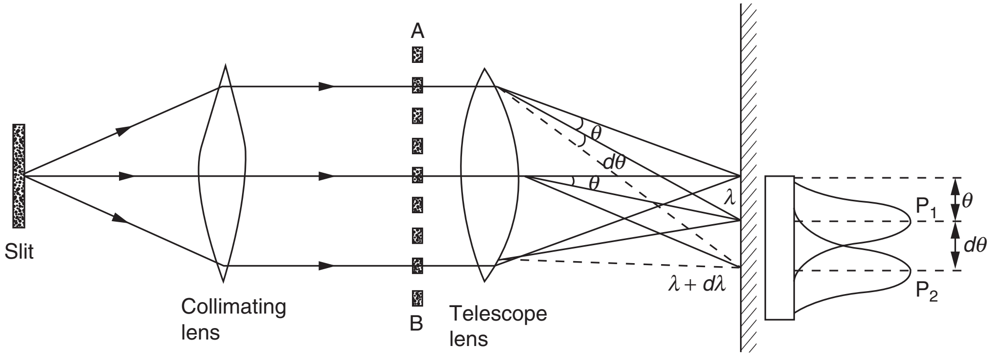


Figure 14 Resolving power of a plane transmission grating.

The direction of (say) nth principal maxima is given by

$$(a+b)\sin\theta = n\lambda$$

The direction of minima is given by

$$N(a+b)\sin\theta = m\lambda$$

where m can take any integer value except $m = 0, N, 2N, 3N, \ldots, nN$ (for nth order mixima). Then the first minima adjacent to this nth principal maxima in θ increasing direction (i.e., $\theta + d\theta$; mind it that on the other side, minima is at $\theta - d\theta$) with $m = nN + 1$ will be obtained by

$$N(a+b)\sin(\theta + d\theta) = (nN+1)\lambda \tag{4.34}$$

Rayleigh's criterion suggests that spectral lines λ and $\lambda + d\lambda$ are just resolved when nth order maxima of $\lambda + d\lambda$ (say) falls on (or overlaps with) the first minima of λ (mind it – adjacent to its nth order maxima) (see in Fig. 14). In other words, for just resolution of spectral lines λ and $\lambda + d\lambda$, the nth maxima of $\lambda + d\lambda$ and first minima of λ (adjacent to its nth maxima) should be formed in the same direction (condition of overlapping), that is, $\theta + d\theta$.

Further, we have nth order maxima of $\lambda + d\lambda$ in $\theta + d\theta$ direction given by

$$(a+b)\sin(\theta + d\theta) = n(\lambda + d\lambda)$$

or

$$N(a+b)\sin(\theta + d\theta) = Nn(\lambda + d\lambda) \tag{4.35}$$

Now, applying Rayleigh's criterion implies nth order maxima of $\lambda + d\lambda$ by Eq. (4.35) and first minima of λ by Eq. (4.34) should overlap, that is, mathematically both Eqs. (4.34) and (4.35) must hold simultaneously. We get

$$(nN+1)\lambda = Nn\ (\lambda + d\lambda)$$

or

$$\lambda/d\lambda = nN = N(a+b)\sin\theta/\lambda$$

Thus the resolving power of grating is the product of the order of spectrum with total number of lines on the grating. Further, we have dispersive power given by

$$d\theta/d\lambda = n/(a+b)\cos\theta$$

Hence,

$$\lambda/d\lambda = nN = N(a+b)\cos\theta\ d\theta/d\lambda = A\ d\theta/d\lambda$$

This implies

$$\text{Resolving Power (RP)} = \text{Aperture} \times \text{Dispersive Power (DP)}$$

Finally, a distinction between RP and DP is that DP is angular separation of two spectral lines while RP is closeness of two spectral lines that can be distinguished.

Solved Examples

Example 1

A light of wavelength 6500 Å falls normally on a slit of width a. For what value of a first minimum for light falls at an angle of 30°?

Solution: We know that the minimum intensity in single slit is

$$a\sin\theta = \pm m\lambda$$

According to the question $\lambda = 6500$ Å $= 6.5 \times 10^{-5}$cm, $\theta = 30°$ and $m = 1$. Hence,

$$a = \left(\frac{m\lambda}{\sin\theta}\right) = \frac{1 \times 6.5 \times 10^{-5}}{\sin 30} = 1.3 \times 10^{-4}\ \text{cm}$$

Example 2

A light of wavelength 6000 Å falls normally on a straight slit of width 0.1 mm. Calculate the total angular width of the central maximum and also the linear width as observed on a screen placed 1 m away.

Solution: We know that the total angular width of the central maxima will be 2θ if the first minimum is formed at an angle θ. Hence, minimum intensity in single slit is

$$a\sin\theta = \pm m\lambda$$

According to the question $\lambda = 6000$ Å $= 6 \times 10^{-5}$cm, $a = 0.1$ mm $= 0.01$ cm and $m = 1$. Hence,

$$\theta = \sin^{-1}\left(\frac{m\lambda}{a}\right) = \frac{1\times 6\times 10^{-5}}{0.01} = 0.344°$$

Total angular width of the central maxima = $2\,\theta = 2\times 0.344° = 0.688°$

Total linear width of central maxima $= \dfrac{2D\lambda}{a} = \dfrac{2\times 100\times 6\times 10^{-5}}{0.01} = 1.2$ cm

Example 3

Find the angular separation between the first-order minima on either side of central maxima when slit is 6×10^{-4} cm wide. Given wavelength of light $\lambda = 6000$ Å.

Solution: We know that the minimum intensity in single slit is

$$a\sin\theta = \pm m\lambda$$

According to the question $\lambda = 6000$ Å $= 6 \times 10^{-5}$ cm, $a = 6 \times 10^{-4}$ cm and $m = 1$. Hence,

$$\theta = \sin^{-1}\left(\frac{m\lambda}{a}\right) = \frac{1\times 6\times 10^{-5}}{6\times 10^{-4}} = \sin^{-1}(0.1) = 5°44'21''$$

Angular separation between the first-order minima $2\theta = 2\times 5°44'21'' = 11°28'42''$.

Example 4

A diffraction grating used at normal incidence gives a green line (5400 Å) in a certain order superimposed on the violet line (4050 Å) of the next higher order. If the angle of diffraction is 30°, how many lines per cm are there in grating?

Solution: We know that the *n*th order principal maxima is

$$(a + b)\sin\theta = n\lambda$$

According to the question, for the green light $\lambda = 5400$ Å , $\theta = 30°$ and the order is n. So

$$(a + b)\sin 30° = 5400\, n \tag{4.36}$$

For the violet light $\lambda = 4050$ Å, $\theta = 30°$ and the order is $n + 1$. So

$$(a + b)\sin 30° = 4050(n + 1) \tag{4.37}$$

From Eqs. (4.36) and (4.37), we have

$$5400n = 4050(n + 1) \Rightarrow n = 3$$

Substituting this value in Eq. (4.36) we get

$$(a + b)\sin 30° = 5400 \times 3(a + b) = 32400 \text{ Å} = 3.24 \times 10^{-4} \text{ cm}$$

$$\text{Number of lines per cm} = \frac{1}{(a+b)} = \frac{1}{3.24\times 10^{-4}} = 3086.42$$

Example 5

How many orders will be visible if the wavelength of incident radiation is 5000 Å and the number of lines on the grating is 2620 per inch?

Solution: We know that the nth order principal maxima is

$$(a+b)\sin\theta = n\lambda$$

According to the question, for maxima $\theta = 90°$, $\lambda = 5000 \text{ Å} = 5 \times 10^{-5}$ cm and $(a+b) = (2.54/2620)$ cm. So

$$n = \frac{(a+b)}{\lambda} = \frac{2.54}{2620 \times 2620} = 19.39$$

Example 6

A diffraction grating has a resolving power $R = \lambda/\Delta\lambda = Mn$. Show that the corresponding frequency range $\nabla\nu$ that can be just resolved is given by

$$\Delta\nu = \frac{c}{Mn\lambda}$$

Solution: The resolving power of two wavelengths with difference $\Delta\lambda$ is

$$R = \frac{\lambda}{\Delta\lambda} = Mn$$

M is total number of lines on grating and n is order of spectrum. If ν is the frequency corresponding to λ and $\Delta\nu$ is the range of frequency corresponding to $\Delta\lambda$ then

$$\nu = \frac{c}{\lambda} \qquad \text{[}c\text{ is velocity of light]}$$

and
$$\Delta\nu = -c\frac{\Delta\lambda}{\lambda^2}$$

The magnitude is given by

$$\Delta\nu = \frac{c}{\lambda}\left(\frac{\Delta\lambda}{\lambda}\right) = \frac{c}{\lambda}\left(\frac{1}{Mn}\right) \qquad \left[\because \frac{\Delta\lambda}{\lambda} = \frac{1}{Mn}\right]$$

or
$$\Delta\nu = \frac{c}{Mn\lambda}$$

Short Answers of Some Important Questions

1. What is diffraction?

Answer: When a beam of light passes through small aperture or a narrow slit, it does not follow rectilinear path but bends rounds corners of the obstacles (slit or aperture). Thus, the bending of light from obstacles (slit or aperture) is known as diffraction.

2. What do you understand by Fraunhofer diffraction?

Answer: In Fraunhofer diffraction, source or screen or both are at infinite distance from obstacles.

3. What is the effect of intensity when single slit is replaced by double slit?

Answer: When single slit is replaced by double slit then the intensity will be four times.

4. What do you understand by resolving power of an optical instrument?

Answer: The ability of an optical instrument to just resolve the images of two nearby point sources/objects is called its resolving power.

5. What is dispersive power of grating?

Answer: The dispersive power of a diffraction grating is defined as the rate of change of angle of diffraction with the wavelength of light.

6. Why is diffraction not observed for light passing through a window in a room?

Answer: The minima and the secondary maxima will come very close to each other and merge, that is why diffraction is not observed for light passing through a window in a room.

7. What is the effect of increasing the wavelength on single slit Fraunhofer diffraction pattern?

Answer: If we increase the wavelength of light, the *n*th order minimum angle increases and as a result, the spectrum becomes wider.

8. Under what situations certain orders in optical gratings are missing or what is missing order in diffraction grating?

Answer: If the maximum intensity due to interference and the minimum intensity due to diffraction term are simultaneously satisfied, those orders are missing from the spectrum.

9. What is Rayleigh criterion for resolution?

Answer: According to Rayleigh, "two very close spectral lines are said to be resolved if the principal maxima in the diffraction pattern of one object or wavelength coincides with the first minimum in the diffraction pattern of the other."

Important Points and Formulas

1. The bending of light from obstacles (slit or aperture) is known as diffraction.
2. The phenomena of diffraction can be classified into two following categories:
   - **Fresnel diffraction:** In Fresnel diffraction, the source or screen or both are at finite distance from obstacles.
   - **Fraunhofer diffraction:** In Fraunhofer diffraction, the source or screen or both are at infinite distance from obstacles.
3. The direction of principal maxima is given by

   $(a+b)\sin\theta_n = \pm n\lambda$ (Interference maxima)
4. The direction of diffraction minima is given by

   $a\sin\theta = \pm m\lambda$ (Diffraction minima)
5. Dispersive power

   $$\frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta}$$
6. The ability of an optical instrument to just resolve the images of two nearby point sources/objects is called its resolving power.

Multiple Choice Questions

1. Bending of light around sharp edges of an object is known as
   (a) scattering
   (b) diffraction
   (c) reflection
   (d) refraction

2. The total path difference depends only on the angle of diffraction in
   (a) Fraunhofer diffraction
   (b) Fresnel diffraction
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)

3. Which one of the following quantity remains constant during diffraction process?
   (a) Speed
   (b) Wavelength
   (c) Frequency
   (d) All the above

4. The distance between slit and screen is finite in
   (a) Fresnel diffraction
   (b) Fraunhofer diffraction
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)

5. In Fraunhofer diffraction, the distance between slit and screen is
   (a) finite **(b)** infinite
   (c) 1.25 cm **(d)** None of these

6. In Fresnel diffraction, the wave front incident on the slit is
   (a) circular
   (b) spherical
   (c) cylindrical
   (d) either cylindrical or spherical

7. The wave front incident on the slit in Fraunhofer diffraction is
   (a) circular **(b)** spherical
   (c) cylindrical **(d)** plane

8. In Fraunhofer single slit diffraction, the path difference between rays from the slit is
   (a) $a\sin\theta$ **(b)** $a\cos\theta$
   (c) $a\sec\theta$ **(d)** $\frac{a}{\sin\theta}$

9. In Fraunhofer single slit diffraction, the phase difference between rays from the slit is
   (a) $\frac{2\pi}{\lambda}a^2\sin\theta$ **(b)** $\frac{2\pi}{\lambda}a\sin\theta$
   (c) $a\sin\theta$ **(d)** $\frac{2\pi}{a}\lambda\sin\theta$

10. In Fraunhofer diffraction at a single slit, the intensity of first secondary maximum is
(a) 9.5% **(b)** 0.45%
(c) 4.5% **(d)** 1.62%

11. In Fraunhofer single slit diffraction, the minimum intensities are formed at angles
(a) $\sin^{-1}\left(\frac{n\lambda}{a}\right)$ **(b)** $\cos^{-1}\left(\frac{n\lambda}{a}\right)$
(c) $\sin^{-1}\left(\frac{a}{n\lambda}\right)$ **(d)** None of these

Short Answer Type Questions

1. Explain the difference between interference and diffraction.
2. What do you understand by single slit diffraction?
3. What is Fresnel diffraction?
4. What are missing orders?
5. What is Rayleigh criterion of resolution?
6. What is resolving power of grating?
7. What is dispersive power of grating?
8. What do you understand by grating element?

Long Answer Type Questions

1. What is diffraction of light? Distinguish between Fresnel and Fraunhofer diffraction.
2. Discuss Fraunhofer diffraction pattern due to a single slit. Find the expression for the width of the central maximum.
3. Derive an expression for the intensity distribution due to Fraunhofer diffraction at a single slit and show that the intensity of the first subsidiary maximum is about 4.5% of that of the principal maximum.
4. Discuss the phenomenon of Fraunhofer diffraction at a single slit and show that the relative intensities of the successive maximum are

$$1:\frac{4}{9\pi^2}:\frac{4}{25\pi^2}:\frac{4}{49\pi^2}:\cdots$$

5. Describe the Fraunhofer diffraction pattern due to a double slit. What are the missing orders?
6. Describe the feature of a double slit Fraunhofer's diffraction pattern. What is the effect of increasing the following:
   (a) Slit width.
   (b) Slit separation.
   (c) Wavelength.
7. What do you understand by missing order spectrum? What particular spectra would be absent if the width of transparencies and opacities of grating are equal?
8. Show that only first order of spectra is possible if the width of a grating element is less than twice the wavelength of light.
9. Explain the formation of spectra by a plane transmission grating. What are its chief characteristics?
10. Show that the rays forming spectrum in a grating suffer minimum deviation when angle of incidence equals the angle of diffraction.
11. Define dispersive power of a grating and obtain an expression for it.
12. Explain Rayleigh criterion of resolution. Define limit of resolution and resolving power.

Numerical Problems

1. In Fraunhofer diffraction due to narrow slit, a screen is placed 2 m away from the lens to obtain the pattern. If the slit width is 0.2 mm and first minima lie 5 mm on either side of the central maximum, find the wavelength of light.

2. Light of wavelength 5000 Å is incident normally on a single slit. The central maximum falls out at 30° on both sides of the direction of the incident light. Calculate the slit width. For what width of the slit the central maximum would spread out to 90° from the direction of the incident light.

3. Plane wave of $\lambda = 6.0 \times 10^{-5}$ cm falls normally on a slit of width 0.20 mm. Calculate (a) the total angular width of the central maximum and (b) the linear width of the central maximum on a screen placed 2 m away.

4. In a single slit diffraction pattern the distance between the first minimum on the right and the first minimum on the left is 5.2 mm. The screen on which the pattern is displaced is 80 cm from the slit and the wavelength is 5460Å. Calculate the slit width.

5. Two parallel slits have widths 0.15 mm each and separation between them is 0.30 mm. They are illuminated normally by light $\lambda = 6000$Å and the emergent light is focused by a convergent lens of 100 cm focal length. Deduce the positions of the first four interference maxima on one side in the focal plane of the lens.

6. A parallel beam of monochromatic light is allowed to be incident normally on a plane grating having 1250 lines per cm and second-order spectral lines are observed to be deviated through 30°. Calculate the wavelength of the spectral line.

Answers

Multiple Choice Questions

1. (b)
2. (a)
3. (d)
4. (a)
5. (b)
6. (d)
7. (d)
8. (a)
9. (b)
10. (c)
11. (a)

Numerical Problems

1. 5000 Å
2. 5000 Å
3. (a) 6.0×10^{-3} cm, (b) 1.2 cm
4. 1.68×10^{-2}cm
5. 1.33 mm, 2.66 mm, 5.32 mm and 6.6 mm
6. 20,000 Å

5 Polarization of Light

LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Phenomena of double refraction.
- Nicol prism.
- Production and analysis of plane.
- Circular and elliptical polarized light.
- Retardation plate.

5.1 Introduction

We have already discussed the phenomenon of interference and diffraction in Chapters 3 and 4 which validates the wave nature of light. Since these phenomenon are exhibited by transverse and longitudinal waves, therefore, they cannot explain the motion of wave. For example, whether the light waves are longitudinal and transverse or whether the vibrations are linear, circular, or elliptical. The drawbacks were explained on the basis of polarization of light. Now the following question arises: What is polarization?

According to the theory of electromagnetism, electric vector E and magnetic field vector H are mutually perpendicular to each other and also perpendicular to its direction of propagation of light. Most of the optical phenomenon can be explained by the electric vector E (also known as light vector) which vibrates in all directions in a plane perpendicular to the direction of propagation. Thus, *the phenomenon of restricting the vibrations of electric vector E of light in a particular plane is called the polarization of light*.

To understand this phenomenon consider a tourmaline crystal cut with its plane parallel to the crystallographic axis and exposed with unpolarized light; only parallel vibrations of light to the crystallographic axis pass through the crystal. This means that emerging light has vibrations in only one plane, that is, said to be plane polarized light. In this case, tourmaline crystal acts as a polarizer.

5.1.1 Unpolarized Light

The light wave that has vibrations in all directions perpendicular to the direction of propagation of light is known as unpolarized light as shown in Fig. 1.

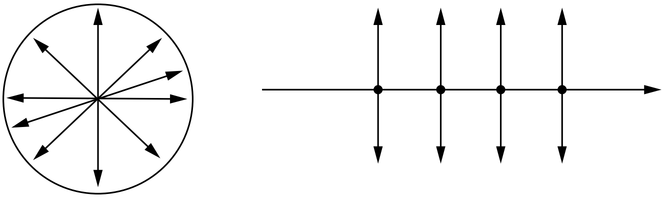


Figure 1 Unpolarized light.

5.1.2 Plane Polarized Light

The light wave that has vibrations in one plane is said to be plane polarized light. In other words, one can say that vibrations along one straight line perpendicular to the direction of propagation of light is called plane polarized light. It is as shown in Fig. 2. In this figure, the arrows represent vibrations confined to the plane of a paper whereas the dots represent the vibrations perpendicular to the plane of a paper.

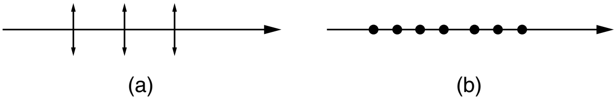


Figure 2 Plane polarized light.

5.1.3 Plane of Vibration

A plane in which the vibrations of polarized light are confined is called plane of vibration of plane polarized light. In Fig. 3, ABCD represents the plane of vibration.

5.1.4 Plane of Polarization

The plane perpendicular to the plane of vibration is called the plane of polarization of plane polarized light. In Fig. 3, PQRS represents the plane of vibration.

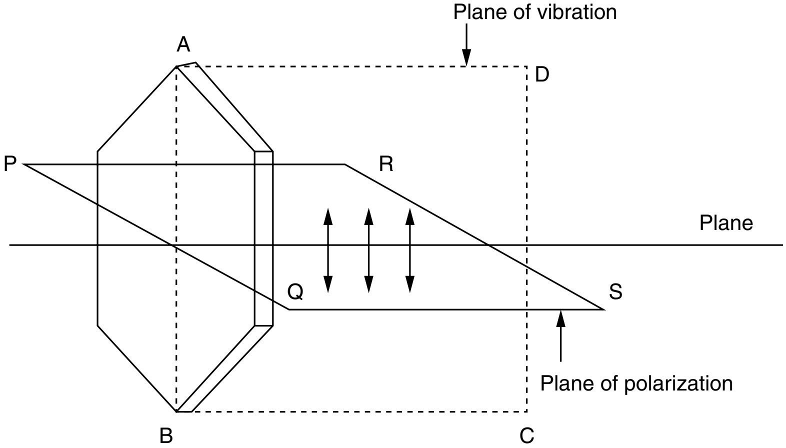


Figure 3 Plane of polarization and plane of vibration.

5.2 Transverse Nature of Light

The phenomenon of polarization confirm the nature of light whether it is transverse or longitudinal. For this purpose, let us take a tourmaline crystal T_1. Cut it with its face parallel to its crystallographic axis. When light beam falls normally on T_1, the emergent beam will be observed to be slightly colored. If this light is allowed to pass another tourmaline crystal T_2 cut similar to T_1, the emergent light shows a variation in intensity as T_2 is rotated [Fig. 4(a)] and when the axes T_1 and T_2 are perpendicular to each other, no light comes out from T_2 [Fig. 4(b)]. These variations in intensities prove the transverse nature of light because longitudinal waves do not undergo polarization.

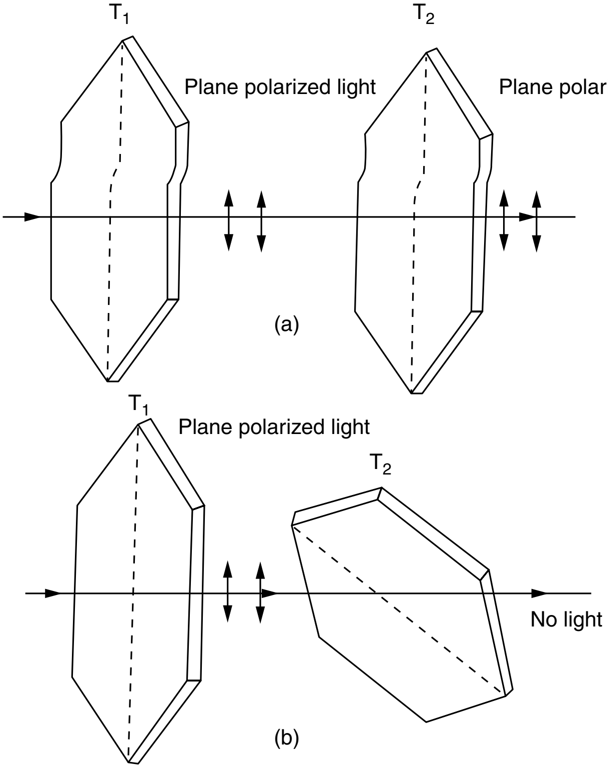


Figure 4 Transverse nature of light.

5.3 Double Refraction and Doubly Refracting Crystals

If unpolarized light or ordinary light pass through glass (or any isotropic medium), the refraction takes place in only one direction whereas if this light is allowed to pass through calcite or tourmaline crystal (i.e., anisotropic medium) then refraction takes place in two directions. One that is everywhere perpendicular to the optic axis and follows all the laws of refraction is called *ordinary ray or O-ray*, whereas the other that is parallel to the optic axis and does not follow the laws of refraction is called *extraordinary or E-ray* as shown in Fig. 5. Thus, *the phenomenon in which incident light ray splits into ordinary or O-ray and extraordinary or E-ray by quartz, calcite or tourmaline crystals is known as double refraction and the crystals which show double refraction are known as doubly refracting crystals.*

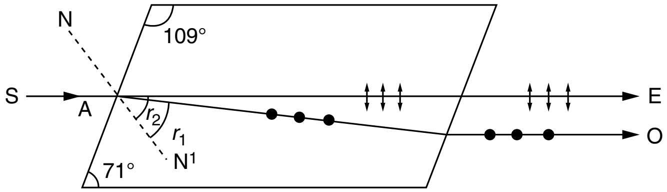


Figure 5 Double refraction.

On the basis of optic axis, there are two types of doubly refracting crystals:

1. **Uniaxial doubly refracting crystals:** In this crystal there is one optic axis along which two refracted light rays travel with the same velocity. Calcite, quartz and tourmaline crystals are examples of uniaxial crystals.
2. **Biaxial doubly refracting crystals:** In this crystal there are two optic axes along which two refracted light rays travel with the same velocity. Mica, topaz and aragonite crystals are examples of biaxial crystals.

5.4 Huygen's Theory of Double Refraction

Huygen explained the phenomenon of double refraction on the basis of secondary wavelets. According to this theory:

1. If the light waves are incident on the surface of uniaxial doubly refracting crystals, each point on the surface of the crystal becomes the origin of two secondary wavelets, O-ray and E-ray.
2. The wavefront of O-ray is spherical because it travels with the same velocity in all directions.
3. The wavefront of E-ray is ellipsoid because it travels with different velocities in different directions.
4. Along optic axis, the velocities of O-ray and E-ray are constant.
5. In negative crystals (calcite) $\mu_0 > \mu_E$, hence $v_0 < v_E$. This means that velocity of E-ray is greater than that of O-ray inside the crystal. That is why the spherical wavefront lies inside the ellipsoid as shown in Fig. 6.
6. In positive crystals (quartz) $\mu_0 < \mu_E$, hence $v_0 > v_E$. This means that velocity of O-ray is greater than that of E-ray inside the crystal. That is why the spherical wavefront lies outside the ellipsoid as shown in Fig. 6.

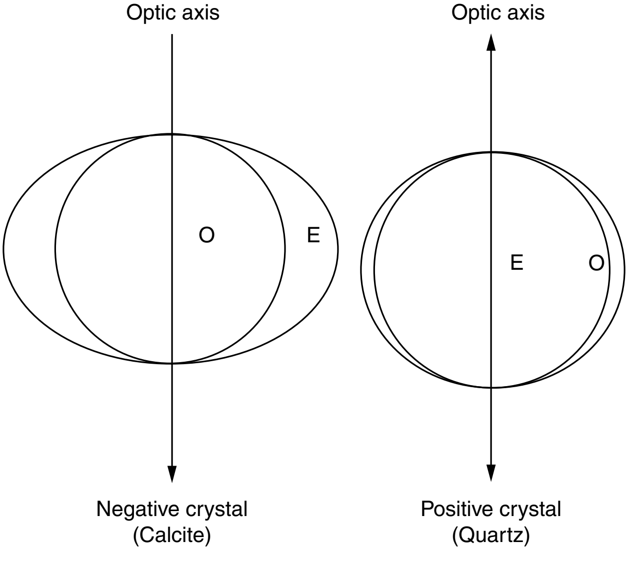


Figure 6 Double refraction on the basis of Huygen's theory.

5.5 Nicol Prism

In 1828, William Nicol invented an optical device which is used for producing and analyzing the plane polarized light.

5.5.1 Principle

The principle of Nicol prism is based on the phenomenon of double refraction. As we know, if unpolarized light passes through calcite crystal, it splits into O-ray which has vibrations perpendicular to the principal section of the crystal and E-ray which has vibrations parallel to the principal section inside the crystal. If somehow one of the two beams is eliminated then only one beam is transmitted through the crystal. In Nicol, O-ray is eliminated by the total internal reflection and E-ray is transmitted through the crystal which is plane polarized light.

5.5.2 Construction

For the construction of Nicol prism, a calcite crystal whose length is three times its width is taken. The two end faces A′D and BC′ of the crystal are cut in such a way that they make an angle of 68° instead of 71°. Resulting part of the crystal is then cut along A′C′ so that it makes an angle 90° with the two end faces as shown in Fig. 7. The two surfaces are grounded, polished optically flat and then cemented together with a transparent material called Canada balsam whose refractive index lies midway between the refractive index of O-ray and E-ray. For sodium light, refractive indices are 1.66, 1.55 and 1.49 for O-ray, Canada balsam and E-ray, respectively.

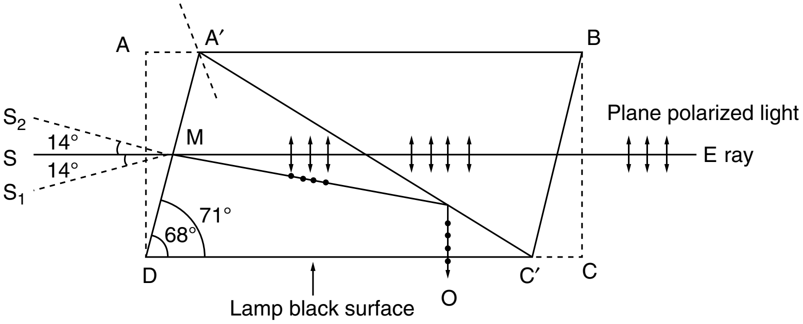


Figure 7 Nicol prism.

5.5.3 Working

If an ordinary light falls on the face A′D parallel to the face DC′, it splits into O-ray and E-ray having vibrations parallel and perpendicular to the principal section inside the crystal. It is clear that Canada balsam layer is more dense than calcite for E-ray and less dense for O-ray (because for O-ray, the angle of incidence at the Canada balsam layer is higher than the critical angle of calcite and Canada balsam), that is

$$\text{Critical angle} = \sin^{-1}(1.55/1.66) = \sin^{-1}(0.933) \approx 69°$$

Therefore, O-ray is reflected from the layer of Canada balsam by total internal reflection and absorbed by the lamp black surface DC′ (see Fig. 7). The E-ray transmitted from Canada balsam layer is plane polarized light. In this way Nicol prism acts as a polarizer.

5.5.4 Nicol Prism as an Analyzer

If two Nicol prisms N_1 and N_2 are parallel to each other then only E-ray passes through both the Nicol prisms. In this case, the first Nicol acts as a polarizer and the other acts as an analyzer as shown in Fig. 8(a). When the second Nicol N_2 is gradually rotated then the intensity of E-ray decreases and if N_1 and N_2 are perpendicular to each other then no light comes out from the second Nicol N_2 [Fig. 8(b)]. Further, if N_2 is rotated, the intensity of emergent light increases. In this way we can say that Nicol prism acts as an analyzer.

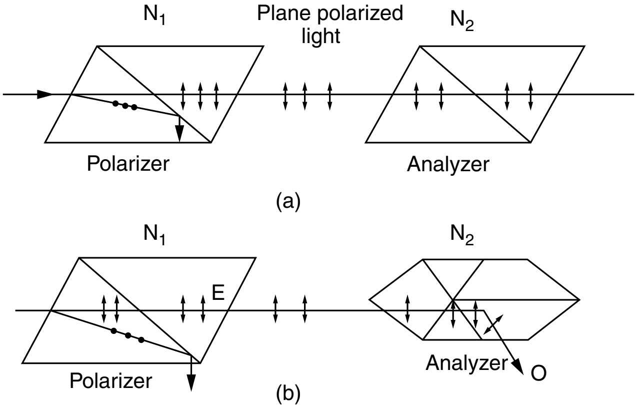


Figure 8 Nicol prism as a polarizer and as an analyzer.

5.6 Mathematical Treatment for Production and Analysis of Plane, Circularly and Elliptical Polarized Light

Let a beam of plane polarized light fall normally on a calcite crystal cut with faces parallel to the optic axis. The vibrations of a beam make an angle θ with the optic axis. The amplitude A of the vibrations is resolved parallel and perpendicular to the optic axis because of doubly refracting calcite crystal. The component parallel to the optic axis is the magnitude of E-ray and the component perpendicular to the optic axis is the magnitude of O-ray as shown in Fig. 9. Since both vibrations (E and O rays) have different velocities in same direction, therefore a phase difference δ is introduced between them. Thus, the displacements of both waves can be written as

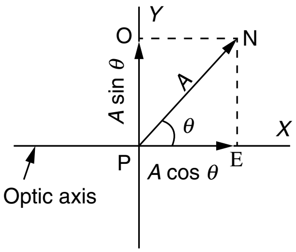


Figure 9 Resolving of amplitude.

$$x = A\cos\theta \sin(\omega t + \delta) \quad \text{for E-ray} \tag{5.1}$$

and

$$y = A\sin\theta \sin\omega t \quad \text{for O-ray} \tag{5.2}$$

Now $a = A\cos\theta$ and $A\sin\theta = b$. So we get

$$x = a\sin(\omega t + \delta) \Rightarrow \frac{x}{a} = \sin\omega t\cos\delta + \cos\omega t\sin\delta \tag{5.3}$$

and

$$y = b\sin\omega t \Rightarrow \frac{y}{b} = \sin\omega t \tag{5.4}$$

From Eqs. (5.3) and (5.4), we have

$$\frac{x}{a} = \frac{y}{b}\cos\delta + \sqrt{1-(y/b)^2}\,\sin\delta \qquad \left(\because\ \cos\omega t = \sqrt{1-\sin^2\omega t} = \sqrt{1-(y/b)^2}\,\sin\delta\right)$$

or

$$\frac{x}{a} - \frac{y}{b}\cos\delta = \sqrt{1-(y/b)^2}\,\sin\delta \tag{5.5}$$

Squaring both sides of Eq. (5.5), we get

$$\left(\frac{x}{a} - \frac{y}{b}\cos\delta\right)^2 = \left(\sqrt{1-(y/b)^2}\,\sin\delta\right)^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos\delta = \sin^2\delta \tag{5.6}$$

Equation (5.6) is general equation of ellipse. The exact nature of the emerging light from the crystal can be obtained by substituting the value of δ in this equation.

Special cases:

1. When $\delta = 0, 2\pi, 4\pi, \ldots, 2n\pi$, then $\cos\delta = 1$ and $\sin\delta = 0$. Therefore Eq. (5.6) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0 \Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0 \quad \text{or} \quad y = \pm\frac{b}{a}x \tag{5.7}$$

   This is the equation of straight line. Thus, the light will be plane polarized [Figs. 10(a) and (e)].

2. When $\delta = \pi, 3\pi, 5\pi, \ldots, (2n+1)\pi$, then $\cos\delta = -1$ and $\sin\delta = 0$. Therefore Eq. (5.6) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0 \Rightarrow \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0 \quad \text{or} \quad y = -\frac{b}{a}x \tag{5.8}$$

   This is again the equation of straight line. Thus, again the light will be plane polarized [Fig. 10(c)].

3. When $\delta = \pi/2, 3\pi/2, 5\pi/2, \ldots, (2n+1)\pi/2$ and $a \neq b$, then $\cos\delta = 0$ and $\sin\delta = 1$. Therefore Eq. (5.6) reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{5.9}$$

   This is the equation of ellipse. Thus, the light will be elliptically polarized [Figs. 10(b) and (d)].

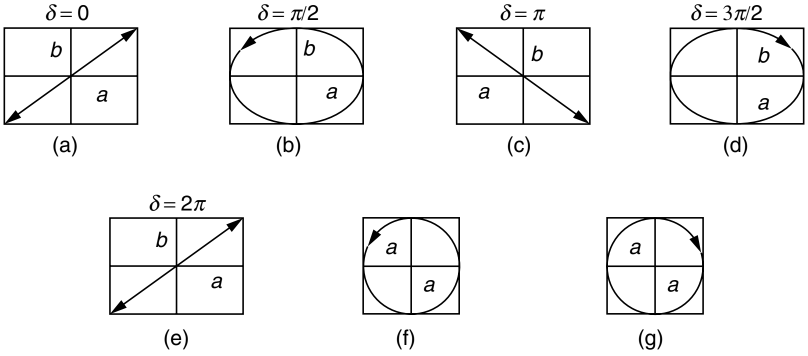


Figure 10 Polarization of vibration.

4. When $\delta = (2n+1)\pi/2$ and $a = b$, then $\cos\delta = 0$ and $\sin\delta = 1$. Therefore Eq. (5.6) reduces to

$$x^2 + y^2 = a^2 \tag{5.10}$$

This is the equation of circle. Thus, the light will be **circularly polarized** when $\theta = 45°$ [Figs. 10(f) and (g)].

5.6.1 Plane and Circularly Polarized Lights are Special Case of Elliptically Polarized Light

From Eq. (5.6), we understand that generally the light coming from the quarter wave plate is elliptically polarized. When $\theta = 0°$ and 90° with the optic axis of quarter wave plate, the transmitted light from the quarter wave plate is plane polarized; however, when $v = 45°$, the transmitted light from the quarter wave plate is circularly polarized. Thus, we can say that plane and circularly polarized lights are special cases of elliptically polarized light.

5.7 Retardation Plates

It is used to produce a phase difference between ordinary and extraordinary rays during the transmission through doubly refracting crystals. There are two types of retardation plates: Quarter-wave and half-wave plates.

5.7.1 Quarter-Wave Plate

It is a uniaxial doubly refracting crystal of suitable thickness cut with its optic axis parallel to the refracting faces so that a phase change of $\pi/2$ or path difference $\lambda/4$ is produced between O-ray and E-ray.

Let μ_0 and μ_E be the refractive index of O-ray and E-ray, respectively. As we have earlier discussed, in a negative (calcite) crystal E-ray travels faster than O-ray, so $\mu_0 > \mu_E$. Therefore, the path difference introduced between O-ray and E-ray is

$$\Delta = (\mu_0 - \mu_E)t$$

The path difference $\lambda/4$ is produced between O-ray and E-ray. Hence

$$\frac{\lambda}{4} = (\mu_0 - \mu_E)t \Rightarrow t = \frac{\lambda}{4(\mu_0 - \mu_E)}$$

For positive (quartz) crystal

$$t = \frac{\lambda}{4\,(\mu_E - \mu_0)}$$

5.7.2 Half-Wave Plate

It is also a uniaxial doubly refracting crystal of suitable thickness cut with its optic axis parallel to the refracting faces so that a phase change of π or path difference $\lambda/2$ is produced between O-ray and E-ray.

Let μ_0 and μ_E be the refractive index of O-ray and E-ray, respectively. As we have earlier discussed, in negative (calcite) crystal E-ray travels faster than O-ray, so $\mu_0 > \mu_E$. Therefore, the path difference introduced between O-ray and E-ray is

$$\Delta = (\mu_0 - \mu_E)t$$

The path difference $\lambda/2$ is produced between O-ray and E-ray. Hence

$$\frac{\lambda}{2} = (\mu_0 - \mu_E)t \Rightarrow t = \frac{\lambda}{2\,(\mu_0 - \mu_E)}$$

For positive (quartz) crystal

$$t = \frac{\lambda}{2\,(\mu_E - \mu_0)}$$

5.8 Production and Analysis of Plane, Circularly and Elliptical Polarized Light

5.8.1 Production of Plane Polarized Light

If unpolarized light passes through a Nicol prism, it splits into O-ray and E-ray. O-ray is reflected by the layer of Canada balsam and E-ray is transmitted by the prism. In this way, the emerging light will be plane polarized light.

5.8.2 Production of Circularly and Elliptically Polarized Light

If the emerging light from a Nicol prism is allowed to fall on a quarter wave plate such that its vibrations make an angle 45° with the optic axis of quarter wave plate, the emergent light will be circularly polarized.

If the plane polarized light normally falls on quarter wave plate such that the vibration of incident light makes an angle other than 45° with the optic axis of quarter wave plate, the emergent light will be elliptically polarized.

5.8.3 Detection of Plane, Circularly and Elliptical Polarized Light

5.8.3.1 Detection of Plane Polarized Light

For detection of plane polarized light, it is made to fall on rotating Nicol. If the intensity of emergent light changes from maximum to zero minimum twice in each rotation, then the emerging light is plane polarized.

5.8.3.2 Detection of Circularly Polarized Light

For detection of circularly polarized light, it is made to fall on rotating Nicol. If the intensity of emergent light remains uniform, then emerging light is either circularly polarized or unpolarized.

To differentiate between the two, the beam is allowed to be incident on a quarter wave plate and then on a rotating Nicol. The variation in intensity is observed from zero to maximum, the emerging light is circularly polarized whereas no variation in intensity, the light is unpolarized.

5.8.3.3 Detection of Elliptical Polarized Light

For detection of elliptical polarized light, it is made to fall on rotating Nicol. If the intensity of emergent light varies from maximum to minimum value but minimum $\neq 0$, then emerging light is either elliptically polarized or partially polarized.

To differentiate between the two, the beam is allowed to be incident on a quarter wave plate and then on a rotating Nicol. Then variation in intensity will be observed from zero to maximum for elliptically polarized light whereas for partially polarized light intensity varies but minimum intensity $\neq 0$.

Solved Examples

Example 1

Calculate the thickness of a quarter wave plate for the wavelength of light of 589 nm and $\mu_E = 1.54$ and $\mu_0 = 1.55$.

Solution: We know that the thickness of quarter wave plate is

$$t = \frac{\lambda}{4(\mu_0 - \mu_E)}$$

Given $\lambda = 589 \text{ nm} = 5.89 \times 10^{-5}$ cm, $\mu_E = 1.54$ and $\mu_0 = 1.55$. Hence,

$$t = \frac{5.89 \times 10^{-5}}{4(1.55 - 1.54)} = 1.47 \times 10^{-4} \text{ cm}$$

Example 2

The values of μ_E and μ_0 for quartz are 1.5508 and 1.5418, respectively. Calculate the phase retardation for $\lambda = 5000$ Å when the plate thickness is 0.032 mm.

Solution: The path difference introduced between O-ray and E-ray for quartz is

$$\Delta = (\mu_E - \mu_0)t$$

The corresponding phase difference or phase retardation is

$$\delta = \frac{2\pi}{\lambda}\Delta = \frac{2\pi}{\lambda}(\mu_E - \mu_0)t$$

Given $\lambda = 5000$ Å $= 5.0 \times 10^{-5}$ cm, $\mu_E = 1.5508$ and $\mu_0 = 1.5418 t = 0.32$ mm $= 0.0032$ cm. The phase retardation is given by

$$\delta = \frac{2\pi}{5.0\times 10^{-5}}(1.5508 - 1.5418)\times 0.0032 = 3.62 \text{ radian}$$

Example 3

Plane polarized light passes through the quartz plate with its axis parallel to its face. Calculate the thickness of the plate so that the emergent light may be plane polarized. For quartz $\mu_E = 1.55$ and $\mu_0 = 1.54$, $\lambda = 589$ nm.

Solution: We know that if plane polarized light passes through a half-wave plate then emerging light will be plane polarized. The thickness of quarter-wave plate is

$$t = 2\frac{\lambda}{4(\mu_E - \mu_0)}$$

Given $\lambda = 589$ nm $= 5.89 \times 10^{-5}$ cm, $\mu_E = 1.55$ and $\mu_0 = 1.54$. Hence,

$$t = \frac{5.89 \times 10^{-5}}{2(1.55 - 1.54)} = 2.84 \times 10^{-4} \text{ cm}$$

Short Answers of Some Important Questions

1. What is polarization?

Answer: The phenomenon of restricting the vibrations of electric vector E of light in a particular plane is called the polarization of light.

2. What do you understand by plane of polarization?

Answer: The plane containing the vibrations as well as directions of waves is known as plane of polarization.

3. What do you understand by double refraction?

Answer: If the incident light ray splits into ordinary (or O-ray) and extraordinary (or E-ray) by quartz, calcite or tourmaline crystals then it is known as double refraction.

4. What is quarter-wave plate?

Answer: It is a uniaxial doubly refracting crystal of suitable thickness cut with its optic axis parallel to the refracting faces so that a phase change of $\pi/2$ or path difference $\lambda/4$ is produced between O-ray and E-ray.

Important Points and Formulas

1. The phenomenon of restricting the vibrations of electric vector E of light in a particular plane is called the polarization of light.
2. The phenomenon in which incident light ray splits into ordinary or O-ray and extraordinary or E-ray by quartz, calcite or tourmaline crystals is known as double refraction and the crystals which show double refraction are known as doubly refracting crystals.
3. An optical device which is used for producing and analyzing the plane polarized light is known as Nicol prism.
4. Principle of Nicol prism is based on the phenomenon of double refraction.
5. Thickness of the quarter wave plate for positive crystal $t = \dfrac{\lambda}{4\,(\mu_E - \mu_0)}$
6. Thickness of the half wave plate for positive crystal
$$t = \frac{\lambda}{2\,(\mu_E - \mu_0)}$$
7. The phase difference or phase retardation
$$\delta = \frac{2\pi}{\lambda}\Delta = \frac{2\pi}{\lambda}(\mu_E - \mu_0)t$$

Multiple Choice Questions

1. The nature of light can be obtained in
   (a) interference **(b)** diffraction
   (c) polarization **(d)** None of these
2. The amplitude of vibrations varies only in the case of
   (a) transverse vibrations
   (b) longitudinal vibrations
   (c) Both (a) and (b)
   (d) None of these
3. The light beam having vibrations along all possible planes perpendicular to the direction of propagation is
   (a) polarized
   (b) unpolarized
   (c) circularly polarized
   (d) elliptically polarized
4. The light having vibrations along a single direction perpendicular to the direction of propagation of light is
   (a) unpolarized
   (b) circularly polarized
   (c) linearly or plane polarized
   (d) elliptically polarized
5. The plane containing the direction of propagation of light but containing no vibrations is
   (a) plane of polarization
   (b) plane of vibrations
   (c) Both (a) and (b)
   (d) Neither (a) nor (b)
6. The ray which obeys the ordinary laws of refraction is
   (a) E-ray
   (b) O-ray
   (c) Both (a) and (b)
   (d) None of these
7. The ray which does not obey the laws of refraction is
   (a) E-ray **(b)** O-ray
   (c) Both (a) and (b) **(d)** None of these
8. The E-ray travels faster as compared to O-ray in
   (a) biaxial crystals
   (b) uniaxial negative crystals
   (c) uniaxial positive crystals
   (d) None of these
9. Which of the following is a uniaxial crystal?
   (a) Topaz **(b)** Copper sulphate
   (c) Mica **(d)** Tourmaline
10. Which of the following is a biaxial crystal?
    (a) Topaz **(b)** Calcite
    (c) Tourmaline **(d)** Quarts

11. The uniaxial negative crystal is
    (a) quartz **(b)** iron oxide
    (c) tourmaline **(d)** mica
12. The uniaxial positive crystal is
    (a) ruby **(b)** tourmaline
    (c) mica **(d)** quartz
13. Nicol prism is based on the principle of
    (a) double refraction
    (b) interference
    (c) total internal reflection
    (d) diffraction
14. Nicol prism act as
    (a) polarizer **(b)** analyzer
    (c) Both (a) and (b) **(d)** Neither (a) nor (b)

Short Answer Type Questions

1. What do you mean by plane-polarized light?
2. What is polarization?
3. What do you understand by unpolarized light?
4. What is plane of vibration?
5. What are doubly refracting crystals?
6. Define optic axis and principle section of a crystal.
7. What is the difference between negative and positive crystals?
8. What is a Nicol prism?

Long Answer Type Questions

1. What do you understand by double refraction? What are ordinary and extra- ordinary rays in a uniaxial crystal?
2. Explain the Huygen's theory of double refraction.
3. Describe the construction and working of Nicol prism.
4. What is the principle of Nicol prism? Explain its working.
5. Explain how Nicol prism acts as a polarizer and analyzer.
6. Explain the production and detection of plane circular and elliptical light.

Numerical Problems

1. Find the thickness of a quarter-wave plate when the wavelength of light is equal to 5890 Å. The refractive index of O- and E-rays are $\mu_0 = 1.55$ and $\mu_E = 1.54$ respectively.
2. Find the thickness of a half-wave plate of quartz for sodium light of wavelength 5893 Å. The ordinary and extraordinary refractive indices for sodium are 1.54 and 1.55, respectively.
3. Calculate the thickness of a calcite plate which would convert plane polarized light into circularly polarized light. The principal refractive indices are $\mu_0 = 1.658$ and $\mu_E = 1.486$ and the wavelength of light used is 5890 Å.
4. A beam of linearly polarized light is changed into circularly polarized light by passing it through a slice of crystal 0.005 cm thick. Calculate the difference in the refractive index of the two rays in the crystal assuming this to be minimum thickness that will produce the effect and that the wavelength is 5×10^{-5} cm.

Answers

Multiple Choice Questions

1. (c)
2. (a)
3. (b)
4. (c)
5. (a)
6. (b)
7. (a)
8. (b)
9. (d)
10. (a)
11. (c)
12. (d)
13. (a)
14. (c)

Numerical Problems

1. 1.47×10^{-3} cm
2. 2.95×10^{-3} cm
3. 8.56×10^{-5}cm
4. 2.5×10^{-3}

6 Laser

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Spontaneous and stimulated emission of radiation.
- Population inversion.
- Concept of 3 and 4 level Lasers.
- Construction and working of Ruby, He–Ne lasers and laser applications.

6.1 Introduction

In the present day, if someone asks you about the perfect coherent source of light, then our answer will be LASER. Laser is a monochromatic, collimated, unidirectional, intense and highly coherent beam of light with small divergence. The word LASER stands for Light Amplification by Stimulated Emission of Radiation. The principle of LASER depends on the phenomenon of stimulated emission. Before we know what stimulated emission is, let us see the development of laser.

The first theoretical concept of laser was given by Einstein in 1917. This theoretical concept was used practically by Townes and his group in 1954 and built MASER which is a device that amplifies microwaves. Thereafter, Schawlow and group extended MASER principle for visible range in 1958. In this context, T. H. Maiman developed the first laser device in 1960 using ruby as an active medium. Hence, it is called Ruby laser. Thus, we conclude that laser is nothing but it is the outgrowth of maser.

6.2 Characteristics of Laser Beam

The characteristics of laser beam are as follows:

1. **Coherence:** The laser beam is completely coherent due to stimulated emission of radiation.
2. **Monochromaticity:** The laser beam is perfectly monochromatic because the emitted light is in a narrow frequency band and so the spread is of the order of a few angstroms.
3. **Unidirectional:** Ordinary light source radiates light in all directions. However, a laser beam emits radiation only in one direction due to the resonant cavity.
4. **High intensity:** The laser beam is highly intense due to concentrated energy in a very small region. If P is the power of the source at a distance r, the intensity is given by

$$I = \frac{P}{4\pi r^2}$$

5. **Brightness:** Light from laser is much more brighter than other ordinary sources; mathematically, it is defined as

$$\text{Brightness}\,(B) = \frac{\text{Power of the source}}{\text{Area} \times \text{Solid angle}}$$

6.3 Concept of Coherence

If the phase difference between the two light beams remains constant with time then beams are said to be coherent. There are two types of coherency: Temporal coherence and spatial or lateral coherence.

If the phase difference between two points along the direction of wave propagation remains constant, the coherence is the called temporal coherence [Fig. 1(a)]. However, if the phase difference between the two points on a plane perpendicular to the direction of propagation remains constant, the coherence is spatial or lateral coherence [Fig. 1(b)].

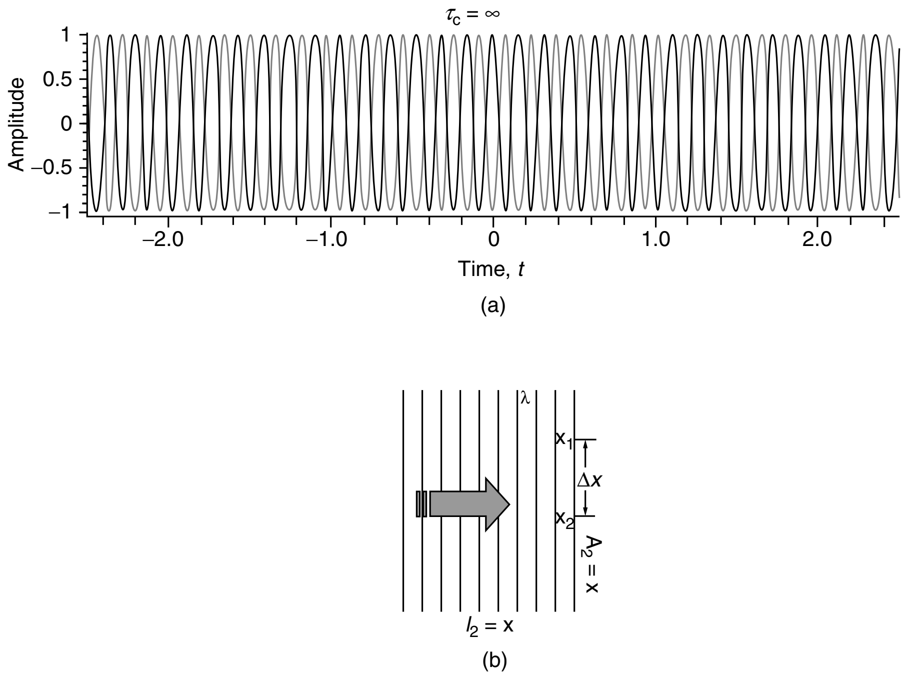


Figure 1 (a) Temporal coherence; (b) spatial coherence.

6.4 Absorption of Radiation

To understand the phenomenon of absorption, let us consider the two energy levels E_1 and E_2 of any atomic system as shown in Fig. 2. If an atom in the ground state of energy E_1 is subjected to incident light radiation, it absorbs the radiation as photon (hv) and jumps to the higher energy state E_2. The photon will be absorbed only if its energy is equal to the energy difference between the excited state E_2 and the ground state E_1

(i.e. $E_2 - E_1$), otherwise photon will not be absorbed and the atom remains in its ground state. *Thus, absorption is the process in which a photon is absorbed by the atom and increasing the number of atoms to its excited state.*

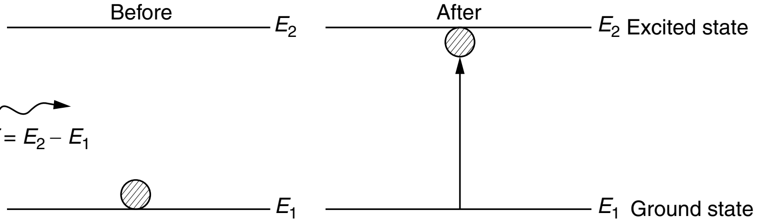


Figure 2 Absorption of radiation.

6.5 Spontaneous Emission of Radiation

When an atom in its excited state E_2 makes a transition to its ground state by emitting a photon of energy $E_2 - E_1$ without any outside influence then it is known as spontaneous emission of radiation.

We are familiar with absorption of radiation through which an atom is excited by photon from ground state E_1 to excited state E_2. But E_2 is not a stable state and after a short interval (10^{-8} sec) of time, the atom jumps back to the ground state by emitting a photon as shown in Fig. 3.

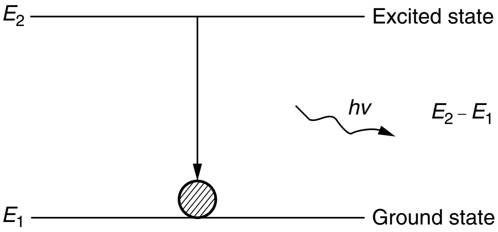


Figure 3 Spontaneous emission of radiation.

The probability of absorption transition $E_1 \rightarrow E_2$ is proportional to the energy density $u(v)$ of radiation of frequency v. Thus, $P_{12} = B_{12}\, u(v)$, where B_{12} is a proportionality constant and known as Einstein coefficient of radiation.

The probability of spontaneous emission $E_2 \rightarrow E_1$ is independent of energy density $u(v)$, and expressed as $P_{21(\text{spontaneous})} = A_{21}$, where A_{21} which is called Einstein coefficient of spontaneous emission of radiation.

Spontaneous emission is incoherent because during spontaneous emission radiation is a random mixture of quanta with various wavelengths. That is why waves neither coincide in wavelength nor in phase.

6.6 Stimulated Emission of Radiation

When an outside photon of energy $hv = E_2 - E_1$ interacts with an atom present in the excited state E_2 as shown in Fig. 4, the photon forces the atom or stimulates it to jump to the lower energy state E_1, emitting an additional photon of same energy.

Thus, stimulated emission is the process by which an atom is induced with photon to make transition from higher energy level to lower energy level by emitting an additional photon of same energy.

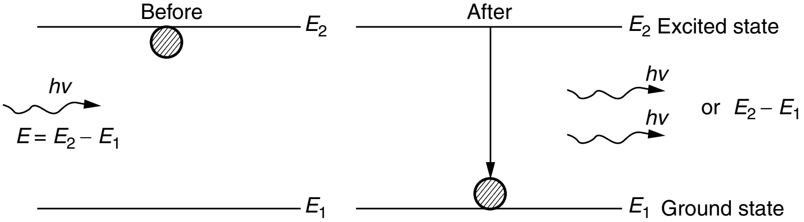


Figure 4 Stimulated emission of radiation.

The probability of stimulated emission transition $E_2 \rightarrow E_1$ is proportional to the energy density $u(v)$ of the stimulated radiation and is given by $P_{21(\text{spontaneous})} = B_{21}\, u(v)$, where B_{21} is the Einstein coefficient of stimulated emission of the radiation.

Thus total probability for transition $2 \rightarrow 1$ is

$$P_{21} = A_{21} + B_{21}\, u(v)$$

The differences between spontaneous and stimulated emission are as follows:

1. In spontaneous emission, transition occurs from higher energy to lower energy without any outside influence. However, in stimulated emission, transition occurs due to induced atom from higher energy to lower energy by emitting photon's energy.
2. Spontaneous emission is incoherent but stimulated emission is coherent.
3. Spontaneous emission is less intense whereas stimulated emission is highly intense.
4. Spontaneous emission has a broad spectrum but stimulated emission has single wavelength.

6.7 Principle of Laser Action

Active system, population inversion, pumping and optical resonator are the main components of laser action. An active system is that where population inversion can be achieved by increasing more atoms in excited state than in ground state. For laser action, this excited state is a metastable state where atom can stay for longer time, about 10^{-3}seconds. This condition is maintained due to continuous supply of energy. When number of atoms in metastable state is higher than lower energy level, the stimulated emission is prompted. Two photons of same frequency are emitted on striking of one photon to excited atom. The emitted photons of same frequency and phase interact with other excited state atoms and produce large number of similar photons. This process is continued and this way a highly intense and unidirectional monochromatic coherent beam of light as laser is produced.

6.7.1 Population Inversion

Population inversion is a necessary phenomenon for the working of laser in which the population of atoms in upper energy state should be more than the ground state. Normally the atoms are available in ground state. When these atoms are given energy by any means they excite to higher energy state and stay there for a short period of 10^{-8} sec. The excited atoms then go to a lower metastable state by radiation of energy and can reside there for a period of 10^{-3} sec which is much larger than the period of stay in higher energy level. This process is continued and the number of atoms in metastable state increases. When the number of atoms in this state is more than the ground state we call it population inversion. The stimulated emission takes place from the metastable state. So the population inversion is *to increase the probability of stimulated emission, the number of atoms in higher energy state (N_2) must be made greater than that of the number of atoms* *in lower energy state* (N_1). The system where population inversion is achieved is known as **active system** and it can be achieved by pumping techniques.

If N_1 and N_2 are the number of atoms in energy states E_1 and E_2, respectively, then relative population inversion in two states is

$$\frac{N_2}{N_1} = e^{-(E_2-E_1)/kT}$$

where k is Boltzmann constant and T is the absolute temperature.

6.7.2 Pumping

The process by which we can raise atoms from lower energy state to higher energy state is known as pumping. Mainly the following four methods are used for the pumping process:

1. **Optical pumping:** In optical pumping the light energy in the form of short flashes is supplied. The ground state atoms absorb this light energy (photons) and excite to higher energy state. This type of pumping used in Ruby laser.
2. **Electrical pumping:** In this method, gas is ionized by means of a suitable potential difference. Strong electric fields accelerate the cathode emitted electrons which collide with the gaseous atoms of the medium. During collision these accelerated electrons give up their energy to gaseous atoms which excite in higher energy state. This type electric pumping is used for population inversion in gas lasers such as He–Ne laser.
3. **Direct electric pumping:** Under this pumping system population inversion is achieved in semiconductor lasers. Here electric current directly creates enough mobile charges (population) at the interphase of two different types of semiconductors. GaAs laser is the example of direct electric pumping.
4. **Chemical pumping:** The exothermic chemical reactions generate enough heat energy to utilize for pumping the atoms to higher energy levels. It can be used in CO_2 laser.

6.8 Various Levels of Laser System

Since two-level pumping is not suitable for obtaining population inversion for laser action, therefore we have introduced three-level and four-level atoms.

6.8.1 Two-Level Laser System

To achieve population inversion in two-level lasers, the number of atoms in excited state E_2 should be higher than ground state E_1. It can be done only if the upper state is a metastable state that is the stay time is greater than 10^{-8} sec. But in two-state system the probability of stimulated absorption is equal to stimulated emission and therefore population inversion cannot be achieved for laser action.

6.8.2 Three-Level Laser System

In this system there are three energy levels E_1, E_2 and E_3 having populations N_1, N_2 and N_3, respectively, introduced for laser action as shown in Fig. 5. According to the figure, the higher energy level E_3 is known as pumping level, lower energy level E_1 is called lower lasing level and the intermediate state E_2 is a metastable state with lifetime 10^{-3} sec. The atoms are raised from E_1 to E_3 by pumping process where after time 10^{-8} sec atoms make a transition from E_3 to E_2 state by non-radiative spontaneous transition and then from E_2 to E_1 which is lasing transition. The problem with three-level laser systems is that it is difficult to produce population inversion because it requires very high pumping power.

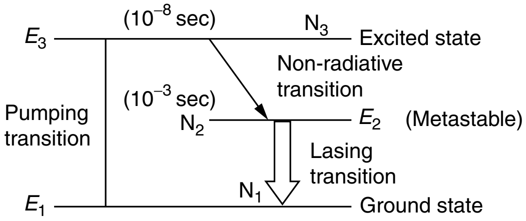


Figure 5 Three-level laser system.

The ruby laser is an example of three-level laser in which ruby crystal is pumped by pulsating light source. It produces a strong pulse of light that is designed to excite the atoms from their ground level to higher level for achieving population inversion.

6.8.3 Four-Level Laser System

In the four-level laser systems, there are four energy levels E_1, E_2, E_3 and E_4 having population of atoms N_1, N_2, N_3 and N_4, respectively, for laser action as shown in Fig. 6. The pumping process takes the atoms from ground state E_1 to excited state E_4. The excited atom makes spontaneous transition from E_4 to lower metastable energy state E_3. The lasing transition is from E_3 to E_2 energy state which is then followed by another spontaneous transition to ground state energy level E_1. He–Ne and Nd–Yag laser are examples of four-level laser systems. The advantage of four-level laser systems is the wideness of exited band and narrowness of metastable band which is required for better lasing action. The only disadvantage of this four-level laser is that more transitions lead to additional loss of energy.

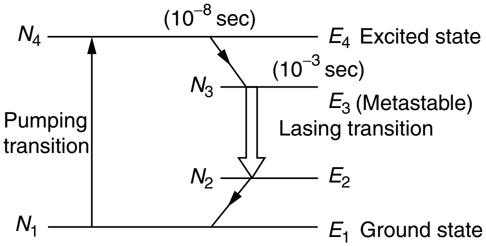


Figure 6 Four-level laser system.

6.9 Ruby Laser

In 1960, Maiman developed the first laser which was based on three energy level system and was called Ruby Laser. Ruby is a crystal of Al_2O_3 (aluminium oxide) doped with a small amount of Cr_2O_3 (chromium oxide). Cr^{+3} ions which are the active material in ruby rod cause the pink color of the crystal and are responsible for laser action. Ruby laser operates in a pulse mode laser beam of wavelength 6943 Å.

6.9.1 Construction

In this laser, there is a ruby rod in the form of a cylindrical rod whose ends are optically flat and parallel. One of the end face is completely silvered and other is only partially silvered. This ruby rod is surrounded by a helical xenon flash tube which provides the pumping light to raise the Cr^{+3} ions to upper energy level as shown in Fig. 7.

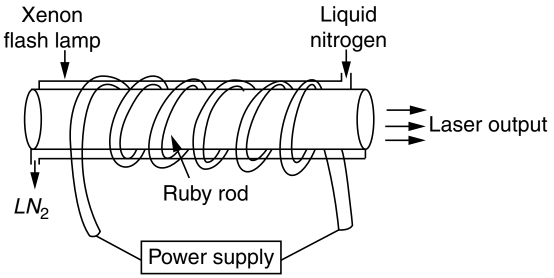


Figure 7 Schematic diagram of ruby laser.

6.9.2 Working

When the flash of light produced by xenon flash tube falls upon the ruby rod, the Cr^{+3} ions in the ground state absorb light of wavelength around 5500 Å (5000–6000 Å) and excited to the upper energy state (see Fig. 8). Thus Cr^{+3} ions are pumped to the excited energy state $E_{3.}$ The xenon flash tube consumes several thousand joules of energy but only a part of this energy is used in pumping the Cr^{+3} ions; rest of the energy heats up the experiment, therefore a proper cooling arrangement is also made.

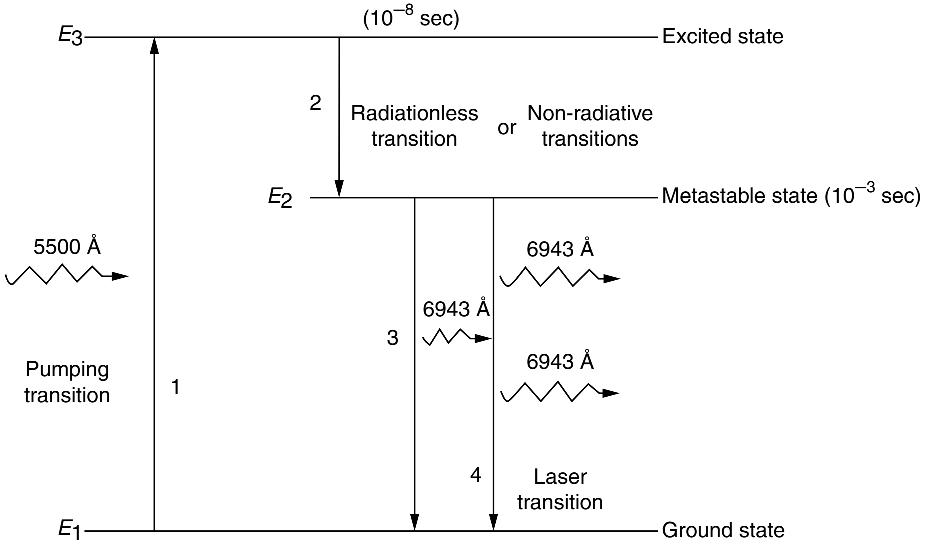


Figure 8 Energy level diagram of ruby laser.

Due to very small mean lifetime of atoms in the excited state (10^{-8} sec) some atoms jump back to the ground state E_1. However some jump to metastable state E_2. Since lifetime of metastable state is in milliseconds (10^{-3} sec) therefore more Cr^{+3} ions are available in this state due to continuous pumping and deficiency pronounced in the ground state. As a result, population inversion is established and then transitions occur from metastable state E_2 to ground state E_1 by emitting a stimulated photon of wavelength 6943 Å.

6.9.3 Drawbacks of Ruby Laser

Some drawbacks of Ruby laser are as follows:

1. Due to three-level laser systems, high power pumping system is required for population inversion.
2. The output beam occurs in the form of microsecond's duration of pulses.
3. The efficiency of Ruby laser is very low because it uses only green part of the pumping light.

6.10 Helium–Neon (He–Ne) Laser

Helium–neon laser is the gas laser based on four-level laser systems. It was successfully operated first time in 1961. It used a mixture of He and Ne in the ratio of 10:1 or 7:1 at a low pressure. This laser is superior to Ruby laser because it gives continuous laser beams of wavelength 6328 Å, 33913 Å and 11523 Å.

6.10.1 Construction

He–Ne laser consists of a glass tube of length 10–100 cm and a narrow diameter of about 2–10 mm. The tube contains the mixture of helium and neon gases. One end of the glass tube is highly silvered whereas other is partially silvered. Two electrodes are inserted in the tube to give high power to achieve population inversion by electric discharge method as shown in Fig. 9.

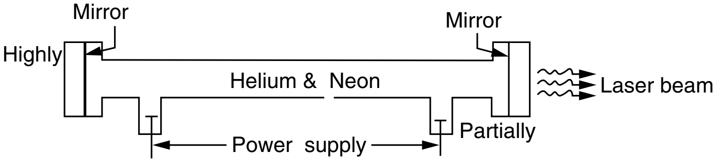


Figure 9 Schematic diagram of He–Ne laser.

6.10.2 Working

The energy level diagram of He–Ne laser is shown in Fig. 10.

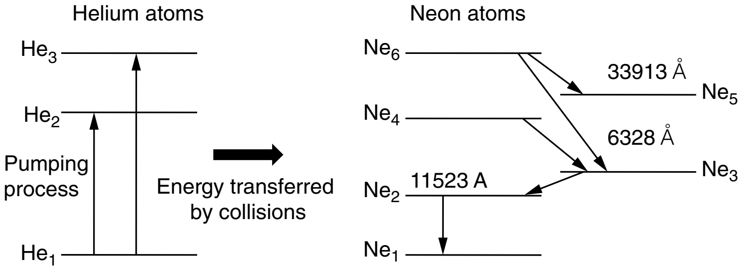


Figure 10 Energy level diagram of He–Ne laser.

A high voltage is applied across the electrodes to ionize the gas. Due to high concentration of the He atoms, the probability of collision of electron and ions with He atoms is higher than that with Ne atom. As a result, He atoms reach the higher energy states. Here some of the excited He atoms collide with Ne atoms and transfer their energy to Ne atoms which excite to Ne_4 and Ne_6 levels to achieve population inversion. This population inversion condition in these levels is maintained at all times due to continuous supply of energy through electric discharge. Ne_2, Ne_4, Ne_6 all are the metastable states for Ne atoms. The transition from Ne_6 to Ne_5 and Ne_3

gives rise to radiations of wavelength 33913 Å and 6328 Å, respectively, whereas transition from Ne_4 to Ne_3 gives radiation of wavelength 11523 Å. The radiation from lower metastable Ne_2 state to the ground state Ne_1 is non-radiative transition. Thus continuous laser beams are obtained in He–Ne laser. He–Ne laser is advantageous and superior to Ruby laser because it produces a continuous beam unlike short pulse in Ruby laser. The gas laser is highly monochromatic, unidirectional, has narrow width of emission and large power output lasers.

6.11 Applications of Laser

Laser has unique properties and is therefore used in various fields of industry, medical, science and engineering. Some of them are described as follows:

1. **Industrial applications of lasers:**
   (a) **Welding:** High power laser is focused on the line of contact to weld the two metal plates. It is used in automobile, ship building, etc. CO_2 and Nd–YAG lasers are used for this purpose.
   (b) **Cutting:** High power lasers are used for cutting the metal sheets. CO_2 or Nd–YAG lasers are also used for this purpose.
   (c) **Hardening:** CO_2 lasers are used for hardening purposes by heating their surface rapidly followed by rapid cooling.
   (d) **Drilling:** High power pulse lasers are used to drill in to the materials. Nd–YAG laser is used for this purpose.
2. **Medical applications:** Lasers are used in laser diagnosis, ophthalmology, bloodless surgeries, dermatology, painless surgeries like cancer treatment, dental, etc. Ruby, CO_2 and Nd–YAG lasers are used for medical applications.
3. **Other fields:** In addition to the above, Ruby, CO_2, He–Ne, and Nd–YAG lasers are used in optical communication, holography, lidar, metrology, etc.

Short Answers of Some Important Questions

1. What do you understand by LASER?

   Answer: The word LASER stands for Light Amplification by Stimulated Emission of Radiation. It is a device which is used to produce monochromatic, collimated, unidirectional, intense and highly coherent beam of light with small divergence.

2. What is the principle of laser?

   Answer: The principle of laser depends on the phenomenon of stimulated emission.

3. What do you mean by coherence?

   Answer: If the phase difference between two beams remains constant then they are said to be coherent.

4. What is absorption of radiation?

   Answer: Absorption is the process of raising the atom to its excited state by which a photon is absorbed by the atom.

5. What do you understand by spontaneous emission?

   Answer: When an atom in its excited state energy state E_2 makes a transition to its ground state by emitting a photon of energy $E_2 - E_1$ without any outside influence, then it is known as spontaneous emission of radiation.

6. Why is spontaneous emission incoherent?

   Answer: Spontaneous emission is incoherent because during spontaneous emission radiation is a random mixture of quanta with various wavelengths. This is why waves neither coincide in wavelength nor in phase.

7. What is stimulated emission?

   Answer: Stimulated emission is the process by which an atom is induced with photon to make transition from higher energy level to lower energy level by emitting an additional photon of same energy.

8. What is population inversion?

Answer: To increase the probability of stimulated emission, the number of atoms in higher energy state (N_2) must be made greater than the number of atoms in lower energy state (N_1). This is called population inversion.

9. What is pumping?

Answer: The process by which we can raise atoms from lower energy state to higher energy state is known as pumping.

Important Points and Formulas

1. The word LASER stands for Light Amplification by Stimulated Emission of Radiation.
2. The laser beam is highly intense due to concentrated energy in a very small region. If P is the power of the source at a distance r, the intensity is given by

$$I = \frac{P}{4\pi r^2}$$

3. Light from laser is much more brighter than other ordinary sources; mathematically, it is defined as

$$\text{Brightness}\,(B) = \frac{\text{Power of the source}}{\text{Area} \times \text{Solid angle}}$$

4. When an atom in its excited state E_2 makes a transition to its ground state by emitting a photon of energy $E_2 - E_1$ without any outside influence then it is known as spontaneous emission of radiation.
5. Stimulated emission is the process by which an atom is induced with photon to make transition from higher energy level to lower energy level by emitting an additional photon of same energy.
6. Two-level pumping is not suitable for obtaining population inversion for laser action,

Multiple Choice Questions

1. Laser is
(a) name of scientist
(b) light absorption by stimulated emission of radiation
(c) light amplification by spontaneous emission of radiation
(d) light amplification by stimulated emission of radiation

2. The emission of photon without any external effect is known as
(a) induced absorption
(b) stimulated emission
(c) spontaneous emission
(d) light amplification

3. The principle of laser is
(a) stimulated emission
(b) spontaneous emission
(c) induced absorption
(d) light amplification

4. Laser produces
(a) monochromatic beam of light
(b) coherent beam of light
(c) low divergence
(d) All the above

5. The population inversion in ruby laser is produced by
(a) electrical discharge
(b) optical pumping

(c) direct conversion
(d) chemical excitation

6. The population inversion in He–Ne laser is produced by
   (a) optical pumping
   (b) photon excitation
   (c) inelastic atomic collisions
   (d) chemical reaction
7. A laser beam consists of
   (a) highly coherent photons
   (b) light material particles
   (c) cosmic rays
   (d) electrons
8. Which of the following is the most stable energy state?
   (a) Metastable state (b) Ground state
   (c) Excited state (d) None of these
9. Which of the following is the least stable energy state?
   (a) Metastable state (b) Ground state
   (c) Excited state (d) All the above
10. The average lifetime of an atom in its metastable state is
    (a) 10^{-8} second (b) 10^{-3} second
    (c) 1 second (d) 10^{-6} second
11. The average lifetime of an atom in excited state is
    (a) 10^{-8} second (b) 10^{-3} second
    (c) 1 second (d) 10^{-6} second
12. The average lifetime of an atom in its ground sate is
    (a) 10^{-8} second (b) 10^{-3} second
    (c) 1 second (d) unlimited
13. The Ruby laser is
    (a) pulsed laser (b) continuous laser
    (c) Both (a) and (b) (d) None of these
14. The He–Ne laser is
    (a) pulsed laser (b) continuous laser
    (c) Both (a) and (b) (d) None of these
15. In He–Ne laser, a mixture of He and Ne gas is in the ratio
    (a) 1:7 (b) 1:10
    (c) 7:1 (d) None of these

Short Answer Type Questions

1. What is a LASER? Who invented the first laser and when?
2. Name the characteristics of laser light.
3. Explain spatial and temporal coherence with suitable diagram.
4. What is pumping?
5. Give the name of different types of pumping.
6. What are the differences between ordinary light and monochromatic light?
7. What is active medium?
8. Why it is easier to achieve the lasing action in a four-level system compared to a three-level system?
9. Why is He–Ne laser superior to Ruby laser?
10. Give some examples of laser.

Long Answer Type Questions

1. What do you mean by LASER? Explain spontaneous and stimulated emission of radiations.
2. What is the principle of laser? Explain the characteristics which differentiate and make laser light useful compared to normal light.
3. What do you understand by pumping? Explain various types of pumping with suitable examples.
4. Explain the basic components of laser system.

5. Describe the construction and working of Ruby laser.

6. Describe the working of He–Ne laser with energy level diagram.

7. Discuss the applications of laser in material.

8. What are the characteristics and applications of laser?

9. Why is He–Ne laser superior to Ruby laser? Explain.

Answers

Multiple Choice Questions

1. (d)
2. (c)
3. (a)
4. (d)
5. (b)
6. (c)
7. (a)
8. (b)
9. (c)
10. (b)
11. (a)
12. (d)
13. (a)
14. (b)
15. (c)

7 Fiber Optics and Holography

LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Fundamental ideas about optical fiber.
- Propagation mechanism.
- Acceptance angle and cone.
- Numerical aperture.
- Single and multimode fibers.
- Basic principle of holography.
- Construction and reconstruction of image on hologram and applications of holography.

7.1 Introduction

Fiber optics is the modern technology for data communication or data transfer. In our traditional electronic communication system, information is transferred using radio waves and microwaves over copper wires or coaxial cables. However, the limited bandwidth of this system restricts the data-carrying capacity and is not sufficient for our modern needs. On the other hand, light waves (frequency is in the range of 10^{15} Hz) guided over a hair-like thin fiber enhance bandwidth tremendously and a large amount of data (information) can be transferred simultaneously. The advantage of optical fiber is that it not only increases the rate of data transfer but also has low signal loss, light weight, no electric short circuiting and has secure transfer of data. Optical communication began in 1880 with the first experimental efforts of Alexander Graham Bell who transmitted his own speech using photo phone. Later by 1960, it was established that light waves could be guided with glass fiber and it was possible to transfer information through optical communication, though it was not practicable due to heavy attenuations of light wave signals. The main components of optical communication are mode (optical fiber) and carrier wave (light waves). After the invention of LASER in 1970 and fabrication of low-loss glass fibers, the attenuation loss decreased and the optical communication became practicable. Today optical fiber has proven to be more versatile than copper wire and it is just a matter of time when glass would replace copper totally. In addition to communication channel, optical fiber is also used in other areas like making sensors for detecting different types of energies, in display systems, photocopying machines, fibro scopes, etc.

Optical fiber is a very thin (as thin as human hair) flexible glass or plastic medium of cylindrical shape which can guide light wave along their length. Light is propagated through fiber along its length due to successive total internal reflection (TIR) from the walls of fiber as shown in Fig. 1. In general, a fiber consists of three coaxial sections: Core, cladding and sheath (protecting jacket).

Core is the innermost light-guiding cylindrical region having diameters in the range of 2–50 μm. It is surrounded by another cylindrical coaxial region called *cladding* whose diameter can be up to 125 μm. The refractive index of cladding is always less than the refractive index of core ($\mu_{\text{core}} > \mu_{\text{cladding}}$) to fulfill the condition of total internal reflection. Hence, cladding restricts the light to travel through core. Now both

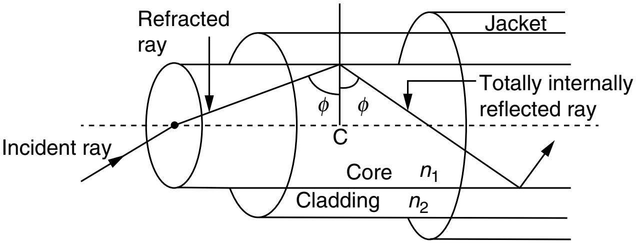


Figure 1 Schematic diagram of optical fiber.

core and cladding are covered by the outermost region called *sheath*. It basically protects the core and cladding from harmful influence of moisture, abrasions and provides the mechanical strength to the fiber. Its diameter is about 150 μm.

7.2 Light Propagation in an Optical Fiber

As we know that light deviates from its original path while travelling from one medium to another. When light travels from denser medium to rarer medium, the refracted ray will go away from the normal as shown in Fig. 2(a).

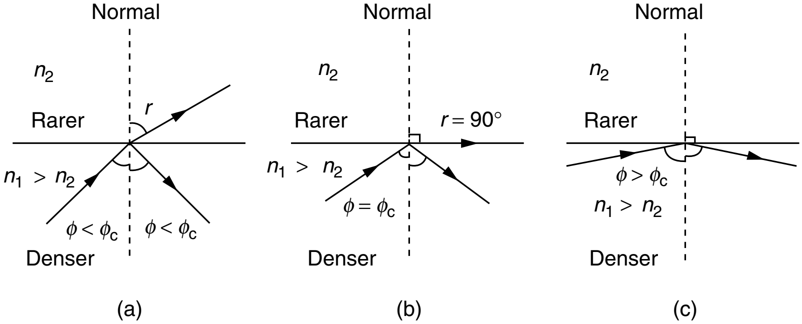


Figure 2 Total internal reflection.

Let us consider the light propagation in an optical fiber. If the light enters the fiber, it follows two necessary conditions for total internal reflection:

1. Light wave should be propagated from denser medium to rarer.
2. The angle of incident should be greater than the angle of incidence.

In the fiber, the refractive index of the core n_1 is greater than that of the refractive index of the cladding n_2 ($n_1 > n_2$). Hence, first condition is satisfied. Therefore, from Snell's law

$$n_1 \sin\phi = n_2 \sin r$$

This means that angle of incident increases the angle of refraction (Fig. 3). If the angle of incident is equal to the critical angle of the core–cladding surface then the light ray makes 90° to the normal as shown in Fig. 2(b). Here $\phi = \phi_c$ and $r = 90°$.

$$\phi_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{7.1}$$

If incident angle is greater than critical angle of core–cladding surface then light ray is reflected in the same medium [Fig. 2(c)]. In this case, the reflected light ray is said to be totally internally reflected and this phenomena is used to propagate the light in optical fiber (Fig. 3).

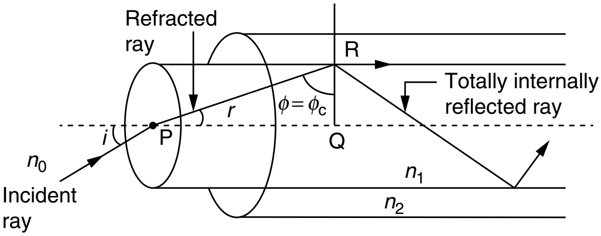


Figure 3 Light propagation in optical fiber using multiple internally reflection.

7.3 Acceptance Angle, Acceptance Cone and Numerical Aperture

7.3.1 Acceptance Angle

As we have already discussed, light signal can be propagated in the fiber using the phenomena of TIR in which incident angle is greater than critical angle of core–cladding surface. In this way, light signal travels in the fiber. Thus, *the maximum angle at which light entering the core is transmitted through the fiber and propagated without being refracted is known as acceptance angle.*

To determine an expression for acceptance angle and numerical aperture, let us consider that the ray of light enters at angle i having refractive index n_0 of the surrounding medium. If light ray is refracted at point P making an angle r with the axis inside the core of refractive index n_1 as shown in Fig. 4, then using Snell's law

$$n_0 \sin i = n_1 \sin r \;\Rightarrow\; \frac{\sin i}{\sin r} = \frac{n_1}{n_0} \tag{7.2}$$

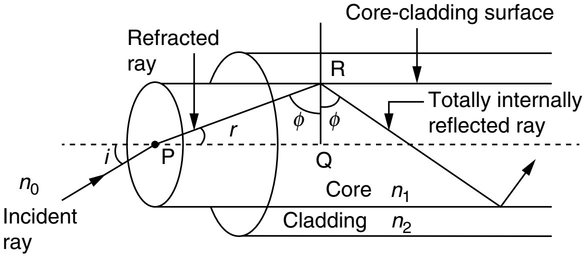


Figure 4 Light propagation in optical fiber.

In Δ PQR

$$r = 90 - \phi \;\text{ or }\; \sin r = \sin(90 - \phi) = \cos\phi \tag{7.3}$$

From Eqs. (7.2) and (7.3), we have

$$\sin i = \frac{n_1}{n_0}\cos\phi$$

For limiting case of total internal reflection $\phi = \phi_c$. Then, $i = i_{\max}$. So

$$\sin i_{\max} = \frac{n_1}{n_0}\cos\phi_c = \frac{n_1}{n_0}\sqrt{1 - \sin^2\phi_c}$$

But $\sin\phi_c = n_2/n_1$. So

$$\sin i_{\max} = \frac{n_1}{n_0}\sqrt{1-\left(\frac{n_2}{n_1}\right)^2} = \frac{\sqrt{(n_1)^2-(n_2)^2}}{n_0} \tag{7.4}$$

For air, $n_0 = 1$. So

$$i_{\max} = \sin^{-1}\left[\frac{\sqrt{(n_1)^2-(n_2)^2}}{n_0}\right] \tag{7.5}$$

This is the required expression for acceptance angle of a fiber.

7.3.2 Acceptance Cone

When the incident ray in launching medium is rotated by 360° about the axis of fiber, it makes a cone which is called acceptance cone. The light coupled with fiber within the acceptance cone is trapped by the core and propagated down the length of fiber (see Fig. 5).

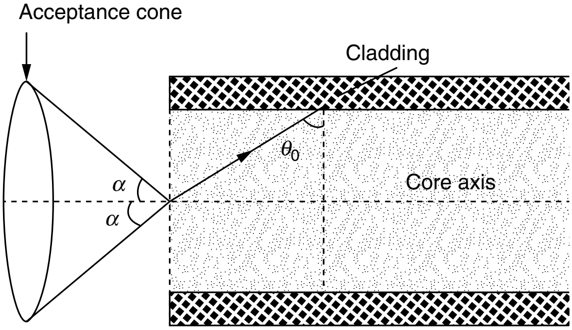


Figure 5 Acceptance cone.

7.3.3 Numerical Aperture

Numerical aperture is an important parameter of the optical fiber. It determines the light gathering ability of fiber, that is, the amount of light that can be accepted by a fiber. It is defined as sine of acceptance angle:

$$\text{NA} = \sin i_{\max} = \sqrt{(n_1)^2-(n_2)^2} = \sqrt{(n_1+n_2)(n_1-n_2)} = \sqrt{2n_1(n_1-n_2)} = \sqrt{2n_1^{\,2}\left(\frac{n_1-n_2}{n_1}\right)} = n_1\sqrt{2\Delta}$$

where Δ is the relative refractive index given by

$$\Delta = \frac{n_1-n_2}{n_1}$$

The low value of NA indicates a small acceptance angle. It is for single-mode step index fiber. A multimode fiber has higher NA which can accommodate number of paths of light. Therefore, a fiber with low NA such as 0.1–0.3 is good for longer distance communications while 0.4–0.5 works well for short distances.

7.4 Modes of Fiber and Normalized Frequency

Light waves are normally propagated through fiber in a zigzag path due to successive reflection from the core–cladding surface if light waves are incident at the surface with an angle greater than critical angle. However all such waves, even if incident at angle greater than critical angle, do not propagate through fiber and only certain ray directions are allowed. These allowed ray paths correspond to the modes of the fiber. So the modes of the fiber are the number of paths of light. These depend on the ratio of diameter of fiber (d) and the wavelength of light (λ). The number of mode that is supported by a fiber is determined from V-number or normalized frequency which is given by

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1^2 - n_2^2}$$

where a is the radius of the core of the optical fiber, λ is the wavelength of the light used and n_1 and n_2 are the refractive index of core and cladding, respectively. The number of modes for step and grade index fibers are $V^2/2$ and $V^2/4$, respectively. For a typical fiber if V is less than 2.405, the fiber is single mode. For $V > 2.405$, it is said to be multimode fiber.

7.5 Types of Fiber

Optical fibers are classified on the basis of modes supported by the fiber and the refractive index profile of the core and cladding of fibers. These are classified into three categories: Single-mode step index, multimode step index and the multimode graded index fibers.

7.5.1 Single-Mode Step Index (SMSI) Fiber

The single-mode step index fiber is a mono-mode that can support only one mode. There is only one path of light or sole ray to travel down the length. The diameter of the core of such fiber is very small, less than about 10 times the wavelength of the propagating light, and can be 2–10 μm (see Fig. 6). If the refractive

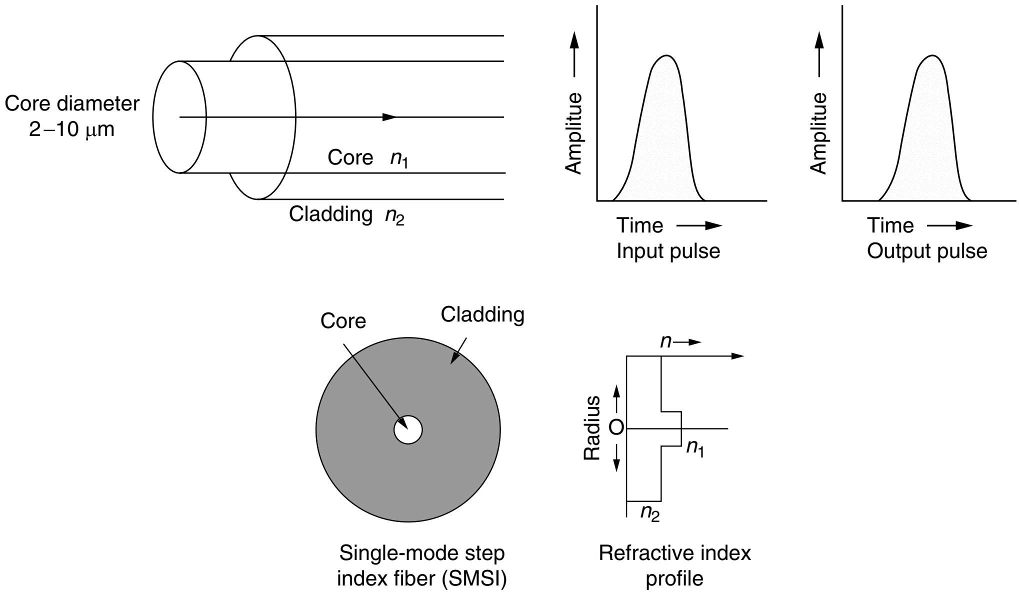


Figure 6 Single-mode step index fiber.

index of fiber changes from core to cladding in steps then it is called step index. There is an abrupt change in the index at core–cladding surface. Being single mode the dispersion losses are very minimum and hence the highest transmission bandwidth. A very high quality, focused, nearly monochromatic light source is used for communication. The numerical aperture and acceptance angle of this fiber are small due to which coupling of light is difficult. Even then because of superior transmission characteristics, these fibers are widely used for long distance communication.

7.5.2 Multimode Step Index (MMSI) Fiber

This fiber is the simplest but poorest in performance. Step index multimode fiber allows more than one mode of wavelength of light to propagate from one end to another as shown in Fig. 7. The core diameter of multimode fiber is in the range 50 μm–200 μm. The refractive index of this fiber, similar to single mode, also abruptly changes at core–cladding surface in steps; hence called step index. Being large core diameter, it can accommodate many rays of light travelling in different paths incident at different angles. Since different modes have different path lengths causing broadening of transmission light pulse, hence dispersion losses are more and the bandwidth length product is small. It is least expensive compared to other fibers but not suitable for large distance communication. So it is used for short distance communication for which signals distortion due to intermodal dispersion is not significant.

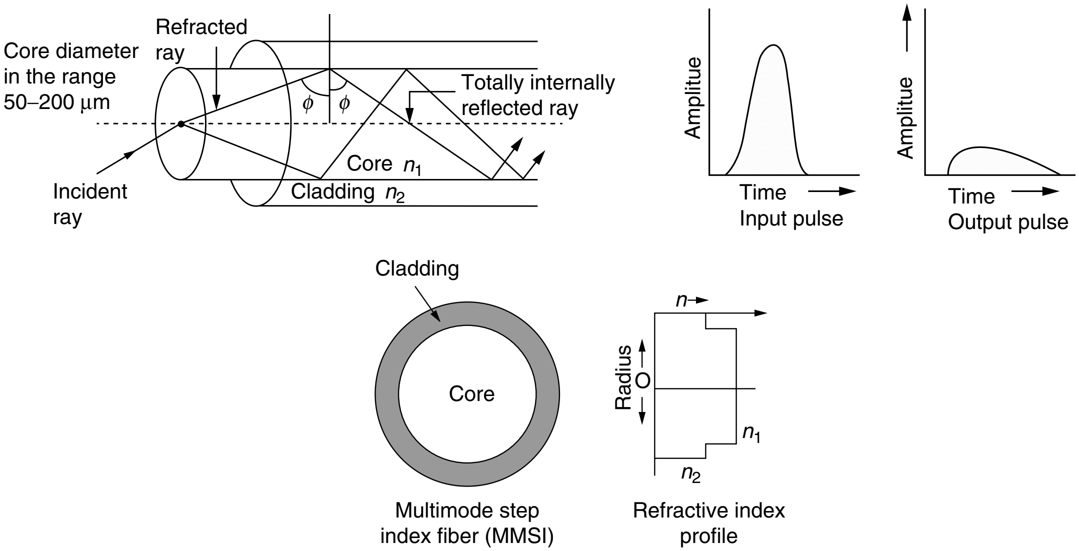


Figure 7 Multimode step index fiber.

7.5.3 Graded Index Optical Fiber

The refractive of the core medium of this fiber is modified in such a way so as to minimize time difference between different modes and hence reduce modal dispersion. The refractive index of the core varies in parabolic form (see Fig. 8). It gradually decreases from the axis of the fiber to the core–cladding surface. When light is incident to the fiber it travels in continuous decreasing refractive index medium from axis to core–cladding surface and in increasing refractive index from core–cladding interface to axis of fiber. Therefore light follows a sinusoidal path instead of straight as in step index fiber. Here the mode having a longer path travels in rare medium with greater speed than a mode having smaller path length in denser medium. Hence, the time difference between modes is reduced which ultimately decreases the modal dispersion. Such fiber can be used for moderate distances.

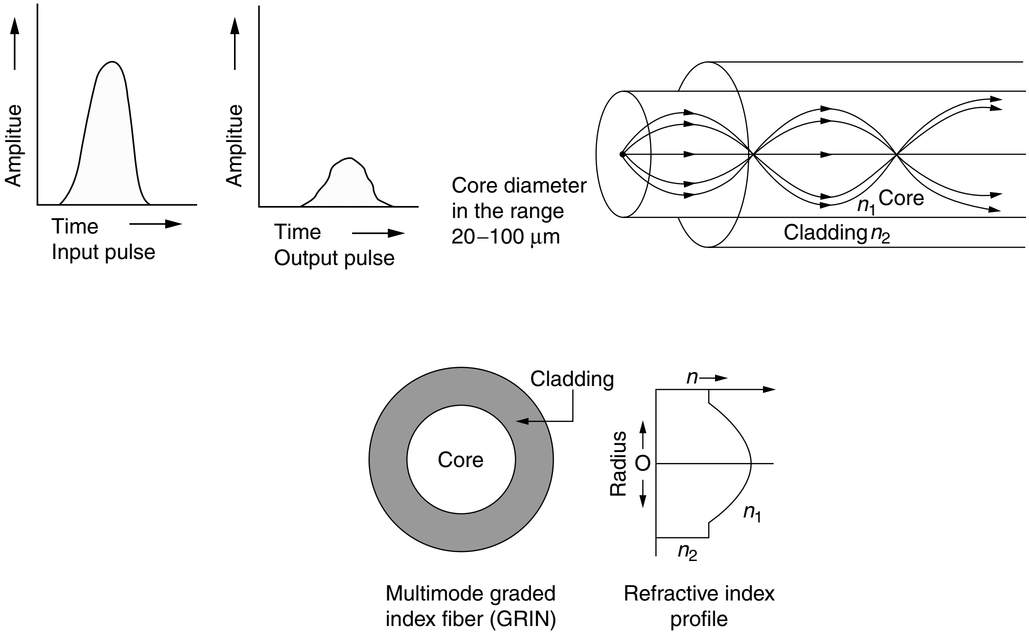


Figure 8 Graded index multimode fiber.

7.6 Comparison of Single-Mode and Multimode Index Fiber

S. No.	*Parameters*	*Single-Mode Fiber*	*Multimode Fiber*
1.	**Mode**	Only one mode is available for propagation of light	More than one mode is available for propagation of light
2.	**Core diameter**	The core diameter is very small	The core diameter is large
3.	**Light source**	Sophisticated light source is used for the propagation of light in a small core	It does not require any sophisticated light source for the propagation of light
4.	**Intermodal dispersion**	It is free from intermodal dispersion	Intermodal dispersion occurs
5.	**Distance communication**	It is used for long distance communication	It is used for short distance communication
6.	**Bandwidth**	Single-mode fiber have higher bandwidth	Low bandwidth
7.	***V*-number**	*V*-number is less than 2.405, $V < 2.405$	*V*-number is greater than 2.405, $V > 2.405$

7.7 Advantages of Optical Fiber Communication

There are many advantages of an optical fiber over others, which are as follows:

1. Optical fibers are not affected by atmospheric conditions; therefore, they have a longer life than copper cables.
2. Due to low transmission losses quality of communication is better.
3. Information carrying capacity of a fiber is higher than that of the other system.
4. Since bandwidth of an optical fiber is higher than that of an equivalent wire, therefore, more channels become available, due to which data transfer speeds are longer.
5. It is cheaper and easier to handle and install.
6. It is smaller in size and lighter in weight.
7. Material cost is lower than that of any other system.
8. It is unaffected by electrical noise and cross-talk.
9. It provides signal security.
10. Attenuation in optical fiber is much lower than coaxial cable.

7.8 Applications of Optical Fiber

Optical fibers are used in various fields such as communication, industry, research, military, medical and in other different appliances. Some of the important applications are given below:

1. First and foremost optical fibers are used in telecommunication field to transfer information from one place to other.
2. They are used in computer applications and broadband applications.
3. Optical fibers are used in industrial applications.
4. They are used in medical fields such as in endoscopy and in angioplasty.
5. Optical fibers are used for undersea transmission cables.
6. They are used in various military applications for a secure information communication.
7. They are used in space applications.
8. Fibers are also used in making different type of sensors.

7.9 Holography

Holography is a technique by which three-dimensional image of an object can be produced. It was invented by a Hungarian physicist Dennis Gabor in 1947 for which he was awarded a Noble Prize in 1971. The coherence properties of laser light are employed to record the image of an object in the form of interference pattern called hologram. The word hologram is the combination of two Greek words 'holos' which means 'complete' and 'gram' which means 'message', the complete message or information. Now, the question arises: What is the difference between the photography and holography? The answer is: The conventional lens photography is a two-dimensional recording of a three-dimensional object. Here the point-to-point recording of amplitude (intensity variation) of the light reflected from object is made on photographic plate as image. The phase information of light in this process is completely lost and hence three-dimensional information is also lost. However, in holography technique, both amplitude as well as phase information are recorded as hologram. Here the information of every part of the object is recorded over the entire area of the hologram. The hologram later produces three-dimensional image through diffraction phenomenon.

7.9.1 Principle of Holography

The principle of holography is based on the phenomena of interference between two beam, that is, object beam and reference beam. The incident laser light is split into two beams. One beam reaches the photographic plate after reflection from the object and is called the object beam. The other beam falls on a plane mirror which reflects it towards the photographic plate and is called reference beam. Both reference and object beams having information of amplitude and phase of the light wave interfere and produce a complete interference pattern. This pattern is recorded on the photographic plate which is called hologram. One can reproduce the three-dimensional image exactly as object from this hologram.

7.9.2 Construction of Hologram

For the construction of image, that is, hologram with holography technique, a laser light is split by a splitter into two beams. One beam is incident on the object whose image is to be formed and other on a plane mirror. The object beam reflected from object towards photographic plate and the reference beam reflected from mirror are the coherent beams and produce an interference pattern which is recorded on the film as shown through Fig. 9. The photographic plate carrying the interference pattern is called hologram. Since this was first invented by Dennis Gabor, therefore, it is also known as Gabor zone plate in honor of Gabor. This hologram basically has the varying intensity pattern which can act as diffraction grating and produce the image of the object by diffraction of light.

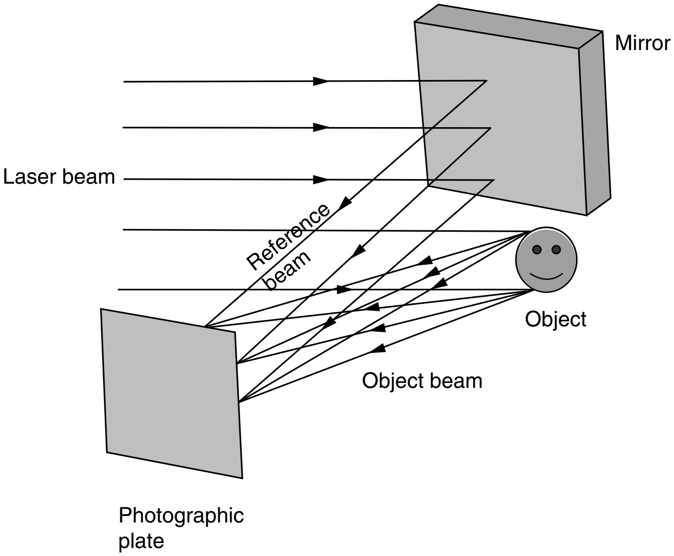


Figure 9 Construction process.

7.9.3 Reconstruction Process

The formation of image from hologram is known as reconstruction of image. In this process, the hologram is exposed by a collimated laser beam which undergoes diffraction process (according to the diffraction theory, when the hologram is exposed with reference beam then hologram behaves like a

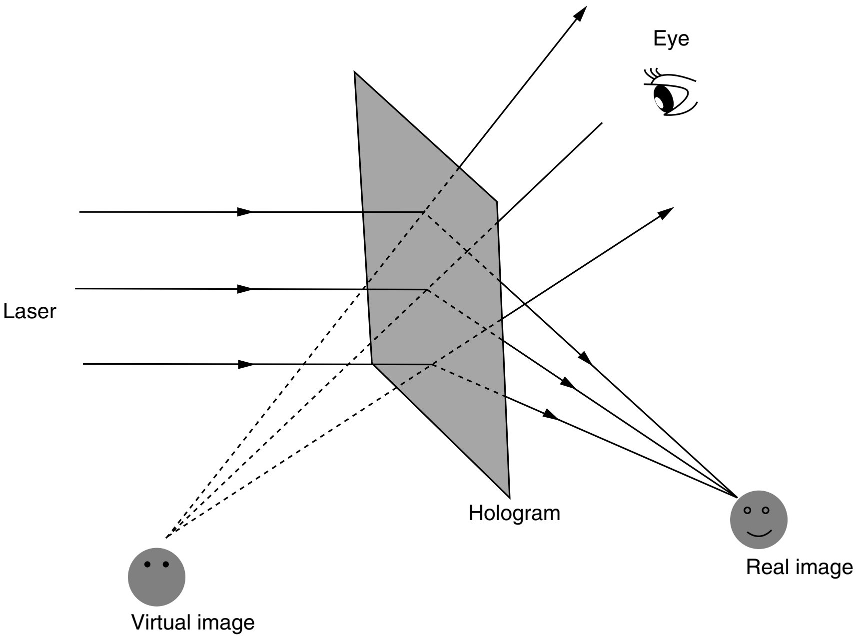


Figure 10 Reconstruction process.

diffraction grating whose every point diffracts the part of the reference beam to reconstruct the wave front from its point source; these individual wave fronts add together to reconstruct the whole of the object beam). One of the diffracted beams emerges from the hologram and provides the virtual image of the object. On the other hand, the second wave converges to form a real image of the object which can be recorded on a photographic plate as shown in Fig. 10. The virtual image has all the characteristics of the object like 3D view which exhibits parallax. That is the different views of image appear by changing the position of the eye.

7.9.4 Characteristics of Hologram

The holograms have some important characteristics as follows:

1. The image constructed by hologram has three-dimensional view and therefore is more realistic than that formed in ordinary photography.
2. In ordinary photography, point-to-point image of an object is recorded on the photographic plate. Therefore, destruction of any part of the image results in complete loss of the information of the object and corresponds to the destructed part. However in holography, the information of every part of the object is recorded on the entire space of hologram. Here every point of the hologram receives light reflected from all parts of the object. Therefore, destruction of any part of hologram does not lose the information of object and it can be retraced from any other part of the hologram. If the hologram is fragmented into different pieces, even a single fragment is enough to reproduce entire image of the object, only with reduced brightness, clarity and definition of the image.

3. In photography only one image can be recorded at one place, however in holography more than one image can be recorded in a single hologram. Here different object beams from various objects at different angles are incident and interfere with same reference beam. Therefore various interference patterns corresponding to the objects are made and recorded at same photographic plate at different angles. Hence the storing capacity of hologram is extremely high. The holography technique allows to store information in a three-dimensional medium at a very high density.

7.9.5 Applications of Holography

Holography is an important tool and has variety of applications in various fields such as industry, science, research, education, etc. Some of the important applications are mentioned below:

1. **Data storage:** Holography can be used to store information at high density at depth inside crystals or photopolymers.
2. **Security:** Holograms are widely used for security purposes on many currencies, credit cards, bank cards as well as DVDs, etc. because it is difficult to forge.
3. **Holographic interferometry:** It is an important technique which enables displacement of objects (static and dynamic) with optically rough surfaces to be measured to optical interferometric precision. With this technique the defects in air craft tyres, automobile engines, artificial bones and quality control are checked.
4. **To detect optical path:** Holography can also be used to detect optical path length variations in transparent media.
5. **Holography in engineering structures:** It can also be used to measure stress, strain and vibration in engineering structures.
6. It is used to study the electric and magnetic field in thin films.
7. It is used to make a hologram of any wave.
8. It is used to determine 3D size of a package in post offices, shipping firms.

Solved Examples

Example 1

A silica glass optical fiber has a core refractive index of 1.5 and cladding refractive index of 1.450. Calculate the numerical aperture of the optical fiber.

Solution: We know that the numerical aperture is

$$\mathrm{NA} = \sin i_{\max} = \sqrt{(n_1)^2 - (n_2)^2} = n_1\sqrt{2\Delta}$$

Here $n_1 = 1.5$ and $n_2 = 1.450$. So the numerical aperture is

$$\mathrm{NA} = \sin i_{\max} = \sqrt{(1.5)^2 - (1.450)^2} = 0.384$$

or

$$\mathrm{NA} = 1.5\sqrt{(2\times 0.033)} = 1.5\times 0.257 = 0.385$$

Example 2

The fraction change of refractive index between the core material and the cladding material of a fiber is 1%. If the refractive index of the core material is 1.46, calculate the numerical aperture and acceptance angle.

Solution: The refractive index of the core material is 1.46 and the fractional difference of the refractive index between the core and cladding material, $\Delta = 1\%$ of $1.46 = 0.0146$. We have $n_1 = 1.46$. The numerical aperture is

$$\text{NA} = n_1\sqrt{2\Delta} = 1.46\sqrt{2\times 0.00146} = 1.46\times 0.171 = 0.25$$

The acceptance angle is

$$i_{\max} = \sin^{-1}(0.25) = 14.5°$$

Example 3

Calculate the numerical aperture, acceptance angle and the critical angle of the fiber from the following data: $n_1 = 1.50$ and $n_2 = 1.45$.

Solution: We know that the numerical aperture is

$$\text{NA} = \sin i_{\max} = \sqrt{(n_1)^2 - (n_2)^2} = n_1\sqrt{2\Delta}$$

Here $n_1 = 1.50$ and $n_2 = 1.45$. So the numerical aperture is

$$\text{NA} = \sin i_{\max} = \sqrt{(1.5)^2 - (1.450)^2} = 0.384$$

or

$$\text{NA} = 1.5\sqrt{(2\times 0.033)} = 1.5\times 0.257 = 0.384$$

The acceptance angle is

$$i_{\max} = \sin^{-1}(0.384) = 22.58°$$

The critical angle is

$$\phi_c = \sin^{-1}\frac{n_2}{n_1} = \sin^{-1}\left(\frac{1.45}{1.50}\right) = 75.16°$$

Example 4

If the fractional difference between the core and cladding refractive indices of the optical fiber is 0.0135 and numerical aperture NA is 0.2425, calculate the refractive indices of core and cladding materials.

Solution: We have $\text{NA} = n_1\sqrt{2\Delta}$. Now

$$\Delta = \frac{n_1 - n_2}{n_1}$$

Given NA = 0.2425 and $\Delta = 0.0135$. So

$$n_1 = \frac{\text{NA}}{\sqrt{2\Delta}} = \frac{0.2425}{0.1643} = 1.48$$

$$\Delta = 0.0135 = \frac{n_1 - n_2}{n_1} = \frac{1.48 - n_2}{1.48}$$

$$\Rightarrow\ 1.48 - n_2 = 0.0135 \times 1.48$$

$$\Rightarrow n_2 = 1.476 - 0.02 = 1.46$$

Example 5

A step index fiber has core refractive index 1.466, cladding refractive index 1.46. If the operating wavelength of the rays is 0.85 μm, calculate the cutioff parameter or normalized frequency and the number of modes which the fiber will support. The diameter off core is 50 μm.

Solution: We know that cut-off parameter or cut-off number or normalized frequency is

$$V = \frac{2\pi a}{\lambda}\sqrt{n_1 - n_2}$$

where a is the radius of the core, n_1 is the refractive index of the core, n_2 is the refractive index of the cladding and λ is the operating wavelength. We have $a = 50/2 = 25\,\mu\text{m}, \lambda = 0.85\,\mu\text{m},\ \mu_1 = 1.466,\ \mu_2 = 1.46.$ So

$$V = \frac{2\times 3.14\times 25}{0.85}\sqrt{(1.466)^2 - (1.46)^2} = 184.70\sqrt{2.149 - 2.131}\ = 184.70\times 0.134 = 24.75$$

The number of modes is given by

$$N = \frac{V^2}{2} = \frac{(24.75)^2}{2} \approx 306$$

Short Answers of Some Important Questions

1. What do you understand by fiber optics?

   Answer: Fiber optics is made of glass or plastic pipes as thin as human hair, through which light can be propagated. It consists of essentially three regions. The innermost region is known as core. It is made of glass or plastic. The middle region which covers the core with another layer of glass or plastic is called the cladding. The outermost region is called jacket. It is also made of plastic or polymers.

2. What is the principle of fiber optics?

   Answer: Fiber optics is based on the principle of total internal reflection (TIR) so that optical signal can travel long distances with negligible loss.

3. Give the necessary conditions for total internal reflection.

   Answer: The necessary conditions for total internal reflection are as follows:

   - Light wave should be propagated from denser medium to rarer.
   - The angle of incident should be greater than critical angle.

4. What is acceptance angle and numerical aperture?

   Answer: The maximum angle at which light entering the core is transmitted through the fiber and propagated without being refracted is known as acceptance angle. The sine of acceptance angle is known as numerical aperture.

5. How is light transmitted through fiber optics?

   Answer: When the light signal is incident on the core and makes an angle of incidence greater than the critical angle of the core–cladding surface, multiple total internal reflection phenomena occurs. In this way, light signals can be propagated or transmitted through the fiber.

6. What do you understand by modes of propagation?

   Answer: When light ray travels along different ray paths in fiber, these paths are known as modes of propagation.

7. Distinguish between holography and photography.

   Answer: A photograph is a two-dimensional recording of a three-dimensional scene; however, holography is a technique of producing three-dimensional image of an object. In photography only amplitude of light coming from the object is recorded on photographic plate whereas in holography, amplitude as well as phase are recorded.

Important Points and Formulas

1. Optical fiber is a very thin (as thin as human hair) flexible glass or plastic medium of cylindrical shape which can guide light waves along their length.
2. Light is propagated through fiber along its length due to successive total internal reflection (TIR) from the walls of fiber.
3. The maximum angle at which light entering the core is transmitted through the fiber and propagated without being refracted is known as acceptance angle.

$$i_{\max} = \sin^{-1}\left[\frac{\sqrt{(n_1)^2 - (n_2)^2}}{n_0}\right]$$

4. Numerical aperture is an important parameter of the optical fiber. It determines the light gathering ability of fiber, that is, the amount of light that can be accepted by a fiber. It defined as sine of acceptance angle.

$$\begin{aligned} \text{NA} = \sin i_{\max} &= \sqrt{(n_1)^2 - (n_2)^2} \\ &= \sqrt{(n_1 + n_2)(n_1 - n_2)} = \sqrt{2n_1(n_1 - n_2)} \\ &= \sqrt{2\,n_1^2\left(\frac{n_1 - n_2}{n_1}\right)} = n_1\sqrt{2\Delta} \end{aligned}$$

Multiple Choice Questions

1. The principle of fiber optics is

   (a) TIR **(b)** reflection
   (c) refraction **(d)** None of these

2. In fiber optics, the core is surrounded by a cladding of

   (a) reflecting medium
   (b) transport medium of lower refraction index
   (c) transport medium of higher refractive index
   (d) opaque medium

3. In which form are telephone signals sent through optical fiber?

   (a) Sound
   (b) Visible or infrared light
   (c) Radio frequency
   (d) All the above

4. In single mode fiber, the diameter of the core is
   (a) 60 μm **(b)** 100 μm
   (c) 10 μm **(d)** 10 m

5. Loss of power during transmission through optical fiber is
   (a) Attenuation **(b)** Modification
   (c) Energy loss **(d)** None of these

6. In optical fiber, numerical aperture is defined as sine of
   (a) core **(b)** cladding
   (c) acceptance angle **(d)** critical angle

7. The best fiber for very long distance communication is
   (a) multimode step index fiber
   (b) single mode step index
   (c) graded index multimode fiber
   (d) None of these

8. Attenuation in fiber optics is due to
   (a) absorption **(b)** scattering
   (c) radiation loss **(d)** All the above

9. Image is stored on a hologram in the form of
   (a) interference pattern
   (b) diffraction pattern
   (c) Both (a) and (b)
   (d) None of these

10. Holography records
    (a) amplitude
    (b) phase
    (c) both amplitude and phase
    (d) None of these

11. Photography means
    (a) amplitude
    (b) phase
    (c) both amplitude and phase
    (d) None of these

12. During reproduction of the image, the hologram acts as
    (a) mirror **(b)** photograph
    (c) diffraction grating **(d)** transparent film

Short Answer Type Questions

1. Explain the basic structure of an optical fiber.
2. Why is refractive index of core higher than that of the cladding?
3. What is the use of the cladding in fiber optics?
4. What is step index fiber?
5. What do you mean by mode of propagation?
6. What is the physical significance of *V*-number?
7. Give the name of different types of optical fiber.
8. What do you understand by dispersion?
9. Distinguish between step index and graded index fiber.
10. Give two applications of optical fiber.
11. What is the principle of optical fiber?

Long Answer Type Questions

1. What is the principle of optical fiber? Explain different types of fiber with suitable diagrams.
2. What do you mean by optical fiber? Explain the propagation mechanism in optical fiber.
3. What is the mode of propagation? Give some examples of fiber optics.
4. Explain acceptance angle, acceptance cone, numerical aperture and critical angle for an optical fiber.
5. What is the principle of holography? Explain its applications.
6. Describe hexagonal close pack structure.
7. What are the characteristics and applications of laser?
8. What do you mean by construction and reconstruction of hologram?
9. Write some applications of holography.

Numerical Problems

1. A silica glass optical fiber has a core refractive index of 1.48 and cladding refractive index of 1.46. Calculate the acceptance angle and numerical aperture of the optical fiber.
2. If the fractional difference between the core and cladding refractive indices of the optical fiber is 0.014 and numerical aperture NA is 0.235, calculate the refractive indices of core and cladding materials. Also calculate the critical angle of the core–cladding surface.
3. A step index fiber has core refractive index 1.47, cladding refractive index 1.46. If the operating wavelength of the rays is 0.85 μm, calculate the cut-off parameter or normalized frequency and the number of modes which the fiber will support. The diameter of the core is 50 μm.

Answers

Multiple Choice Questions

1. (a)
2. (b)
3. (d)
4. (c)
5. (a)
6. (c)
7. (b)
8. (d)
9. (a)
10. (c)
11. (a)
12. (c)

Numerical Problems

1. 14°, 0.2425
2. 1.41 and 1.39, $\phi_c = \sin^{-1}\left(\dfrac{n_2}{n_1}\right) = \sin^{-1}\left(\dfrac{1.39}{1.41}\right)$
3. 24.75 and 306

8 Crystal Structure

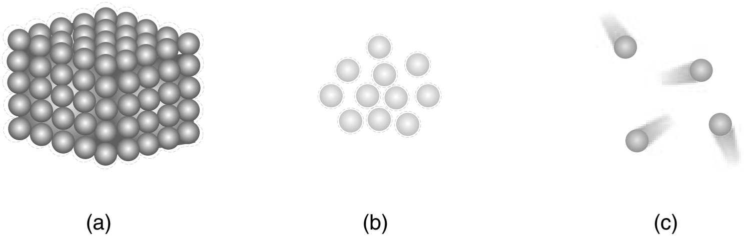
LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Space lattice.
- Basis.
- Unit cell.
- Lattice parameter.
- Seven crystal systems and fourteen Bravais lattices.
- Crystal system structure.
- Packing factor (cubic, body and face).
- Crystal structure of NaCl and diamond.
- Lattice planes and Miller indices.
- Reciprocal lattice.
- Diffraction of X-rays by crystal.
- Laue's experiment.
- Bragg's Law.
- Bragg's spectrometer.

8.1 Introduction

We have already studied in earlier classes that elements and their chemical compounds are present in the form of solids, liquids and gases. All these materials consist of atoms or molecules. In solids and liquids, an inter-atomic distance is of the order of a few Angstrom; however, in the case of gases it is approximately 30 Å, which is reflected through Fig. 1.



(a) (b) (c)

Figure 1 Arrangements of atoms in (a) solid, (b) liquid and (c) gas crystals.

A perfect crystal is considered to be constructed by the infinite regular repetition of identical structural units in space. *Thus, a crystal is a solid in which it is possible to choose a small group of atoms or molecules which can be assumed to be considered in a regular-sided 'box' or unit cell.* There are three different types of solids: Crystalline, polycrystalline and amorphous. Each type is characterized by the size of an ordered region within the material.

8.1.1 Crystalline Solids

In crystalline solids, the atoms or molecules are arranged in regular repeated pattern. So, these materials have long-range orders in their structures [Fig. 2(a)]. Rock salts, quartz and metals are some examples of crystalline solids.

8.1.2 Polycrystalline Solids

In polycrystalline solids, the atoms or molecules are arranged in random directions. Polycrystalline materials have a high degree of order over many atomic or molecular dimensions. These ordered regions, or single-crystal regions, vary in size and orientation with respect to one another [Fig. 2(b)]. Ceramics is an example of polycrystalline.

8.1.3 Amorphous Solids

Amorphous solids are those which lack the regular arrangement of atoms or molecules. So, these materials have short-range orders in their structure [Fig. 2(**c**)]. Glass, silicon are examples of amorphous solids.

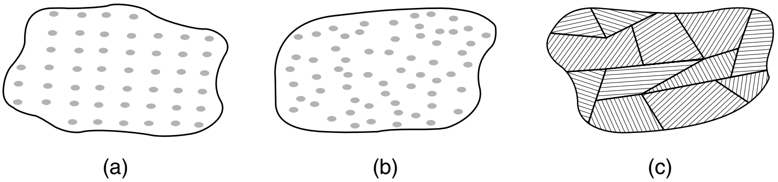


Figure 2 Structure of solids: (a) Crystalline, (b) amorphous, (c) polycrystalline.

8.2 Space Lattice or Crystal Lattice

The periodic arrangement of atoms in a crystal is called a lattice. In other words, lattice is a regular periodic arrangement of points in space that look like a net structure as shown in Fig. 3. *The points where the atoms are situated in the lattice are known as lattice points and the space with these points is known as space lattice.*

In other words, an array of points where an environment around any one point is identical with that of any other point is called plane lattice. If plane lattice is studied in three dimensions then it is said to be space lattice.

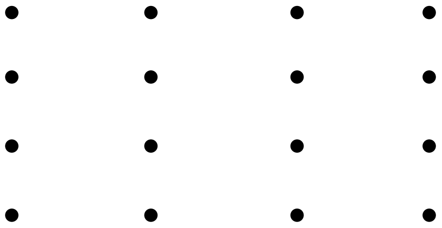


Figure 3 Space lattice.

8.3 Crystal Translational Vectors

The position vector of any lattice point in two-dimensional lattices by choosing any other lattice point as origin is represented by

$$\vec{T} = n_1\vec{a} + n_2\vec{b}$$

where n_1 and n_2 are integers and $\vec{a}, \vec{b}$ are vectors and are called the primitives. For a three-dimensional crystal structure, the arrangements of points can similarly be represented by translational or lattice vector

$$\vec{T} = n_1\vec{a} + n_2\vec{b} + n_3\vec{c}$$

where $\vec{a}, \vec{b}, \vec{c}$ are translational vectors or basis vectors called the primitives along x-, y-, z-axis, respectively, as shown in Fig. 4.

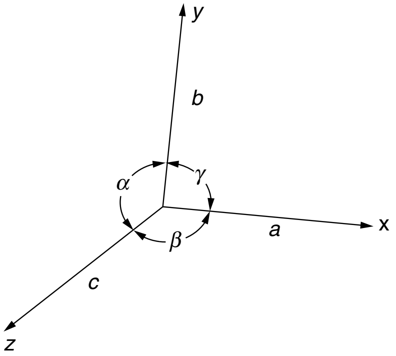


Figure 4 Translational vectors.

8.4 Unit Cells

We shall now discuss further the periodicity of structure which is the fundamental feature of a crystal. Let us consider part of a two-dimensional crystal, atoms of which are arranged in a pattern as shown in Fig. 5. It is observed that when a parallelogram such as ABCD is repeatedly translated by the vectors $\vec{a}$ and $\vec{b}$ corresponding to AB and AD, respectively, to obtain the whole pattern of crystal lattice then fundamental unit ABCD is called a unit cell. Similarly A′B′C′D′ is also a unit cell. *In other words, we can say that the area of the parallelogram whose sides are the basis vectors $\vec{a}$ and $\vec{b}$ is called a unit cell of a lattice.*

In three dimensions, the parallelepiped formed by using the basis vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ *as concurrent edges* is called the unit cell of the space lattice (see Fig. 4). The angles formed between the primitives $\vec{b}$ and $\vec{c}$, $\vec{c}$ and $\vec{a}$, $\vec{a}$ and $\vec{b}$ are denoted by α, β and γ, respectively.

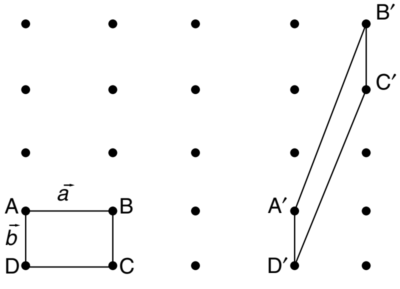


Figure 5 Unit cell.

8.5 Lattice Parameters

Unit cell is nothing but the fundamental elementary pattern of a number of lattice points. For example, the number of lattice points per unit cell for simple cubic (SC), body-centered cubic (BCC) and face-centered cubic (FCC) are one, two and four, respectively. We know that unit cell may not be complete without having a quantitative estimate of its volume which is calculated through the following relation:

$$V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

were V stands for the volume of the cell; **a**, **b** and **c**, defined so far as the measure of unit cell edges are known as *lattice parameters*.

8.6 Density of an Element in terms of Lattice Parameter or Lattice Constant

Let us consider a cubic lattice with each side of the cube as a. If V is the volume of the unit cell and ρ is the density of the crystal, then the volume V of each unit cell is a^3 and the mass of the each unit cell is

$$\text{Mass of unit cell} = \text{Volume} \times \text{Density} = a^3\rho \tag{8.1}$$

If the molecular weight of the material is M and N is the Avogadro number, then

$$\text{Mass of each molecule} = \frac{M}{N}$$

Now if the number of molecules per unit cell is n, then mass of n atoms in unit cell will be

$$\text{Mass} = n \times \frac{M}{N} \tag{8.2}$$

From Eqs. (8.1) and (8.2), we get

$$a^3\rho = n \times \frac{M}{N} \quad \text{or} \quad \rho = \frac{nM}{Na^3}$$

The lattice constant a is

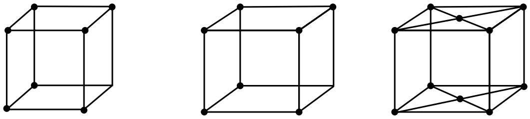
$$a = \left(\frac{nM}{N\rho}\right)^{1/3}$$

8.7 Seven Crystal Systems

Crystals are classified into seven types of crystal systems depending on their axial ratio (*a : b: c*) and angles between them. These differ on the basis of size of unit cell, number of nearest atoms, angular inclinations between the edges of unit cell. The various types are as follows:

1. **Triclinic:** This is the most general type of lattice in which all the translational vectors are unequal and all the axes are not mutually perpendicular. That is $a \neq b \neq c$ and $\alpha \neq \beta \neq \gamma \neq \pi/2$. It is the simple lattice (P) and is shown in Fig. 6.
2. **Monoclinic:** In monoclinic crystals, all the three translation vectors or sides are unequal, with one axis perpendicular to the other two. That is $a \neq b \neq c$ and $\alpha = \beta = \pi/2 \neq \gamma$. There are two types of monoclinic crystal: simple and base-centered as shown in Fig. 6.

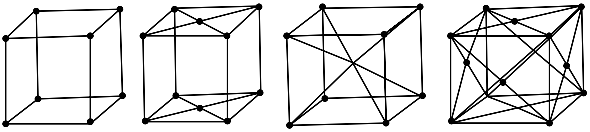
3. **Orthorhombic:** In this crystal, all the sides are unequal, but all the angles are equal, each equal to $\pi/2$. That is $a \neq b \neq c$ and $\alpha = \beta = \gamma = \pi/2$. There are four types of orthorhombic crystals: Simple, base-centered, body-centered and face-centered as shown in Fig. 6.



Simple Triclinic-P

Simple Monoclinic-P

Base-centred Monoclinic-B

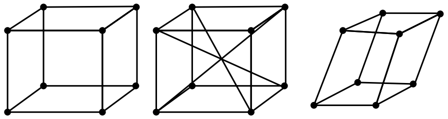


Simple Orthorhombic-P

Base-centered Orthorhombic-B

Body-centered Orthorhombic-I

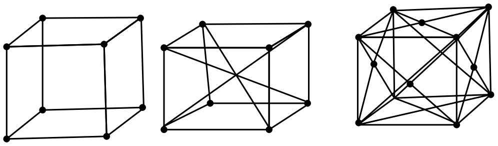
Face-centered Orthorhombic-F



Simple Tetragonal-P

Body-centered Tetragonal-I

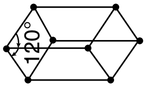
Trigonal-R



Simple Cubic-P

Body-centered Cubic-I

Face-centered Cubic-F



Hexagonal-P

Figure 6 Seven crystal systems.

4. **Tetragonal:** In this crystal, two sides are equal and all the angles are mutually perpendicular. That is $a = b \neq c$ and $\alpha = \beta = \gamma = \pi/2$. There are two types of tetragonal crystals: Simple and body-centered as shown in Fig. 6.
5. **Cubic:** In this crystal, all the sides are equal and all the angles are equal to $\pi/2$. That is $a = b = c$ and $\alpha = \beta = \gamma = \pi/2$. There are three types of cubic crystals: Simple, body-centered and face-centered as shown in Fig. 6.
6. **Trigonal:** In this crystal, all the sides are equal and all the angles are equal and less than $2\pi/3$ but not equal to $\pi/2$. That is $a = b = c$ and $\alpha = \beta = \gamma \neq \pi/2$, but less than $2\pi/3$. If all the angles are equal to $\pi/2$ then it will be cubic lattice as shown in Fig. 6.
7. **Hexagonal:** In this crystal, two sides are equal and inclined at $2\pi/3$ and the third side is perpendicular to both of them (see Fig. 6). That is $a = b \neq c$ and $\alpha = \beta = \pi/2$, and $\gamma = 2\pi/3$.

8.8 Bravais Lattices

Bravais, in 1948, stated that there are 14 possible ways to arrange atoms in a unit cell under the seven crystal systems. These are frequently called the Bravais lattices. The seven crystal systems together with the essential symmetry elements are summarized in Table 1.

Table 1 Seven crystal systems together with the essential symmetry elements

S. No.	*Name of System*	*Nature of Unit Cell Axes and Angles*	*No. of Lattices in Systems*	*Examples*
1.	Triclinic	$a \neq b \neq c$ and $\alpha \neq \beta \neq \gamma \neq \frac{\pi}{2}$	1 (P)	$CuSO_4$, $K_2Cr_2O_7$
2.	Monoclinic	$a \neq b \neq c$ and $\alpha = \beta = \frac{\pi}{2} \neq \gamma$	2 (PB)	$FeSO_{4,}$ Na_2SO_4
3.	Orthorhombic	$a \neq b \neq c$ and $\alpha = \beta = \gamma = \frac{\pi}{2}$	4 (PBIF)	$BaSO_{4,}$ KNO_3
4.	Tetragonal	$a = b \neq c$ and $\alpha = \beta = \gamma = \frac{\pi}{2}$	2 (PI)	$NiSO_4$, TiO_2
5.	Cubic	$a = b = c$ and $\alpha = \beta = \gamma = \frac{\pi}{2}$	3 (PIF)	NaCl, Cu
6.	Trigonal	$a = b = c$ and $\alpha = \beta = \gamma \neq \frac{\pi}{2}$	1 (R)	Sb, $CaSO_4$, Bi
7.	Hexagonal	$a = b \neq c$ and $\alpha = \beta = \frac{\pi}{2}, \gamma = \frac{2\pi}{3}$	1 (P)	Cd, Mg, Quartz
			Total no. of lattices = 14	

Here P is primitive or simple; B is base-centered; I is body-centered; F is face centered.

Cubic system: There are three types of cubic lattices which are as follows:

1. **Simple cubic or P type lattice:** In simple cubic crystal there are eight lattice points at the eight corners of the unit cell. There is no lattice point inside the unit cell [see Fig. 7(a)].
2. **Body-centered or I lattice:** In body-centered cubic crystal there is one additional lattice point situated at the center that belongs exclusively to each cell besides the eight situated at the corners as shown in Fig. 7(b).

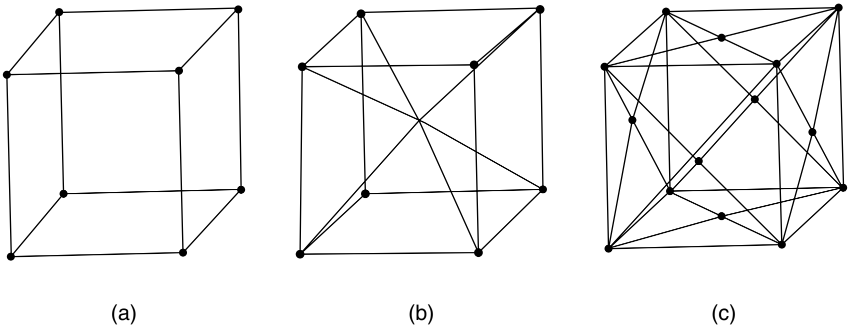

(a) (b) (c)

Figure 7 Cubic systems: (a) Simple, (b) body-centered, (c) face-centered.

3. **Face-centered cubic or F lattice:** In face-centered cubic crystal there are six lattice points situated at the center of each face in addition to those situated at the corners as shown in Fig. 7**(c)**.

8.9 Atomic Radius

Atomic radius is half the distance between the nearest neighbors which are supposed to be in contact with each other along the line of their bonding direction. Atomic radius is represented by *r* and expressed in terms of cube edge *a*.

1. For simple cubic structure, atoms touch each other along the lattice as shown in Fig. 8. Therefore

$$r + r = a \text{ or } 2r = a \Rightarrow r = a/2$$

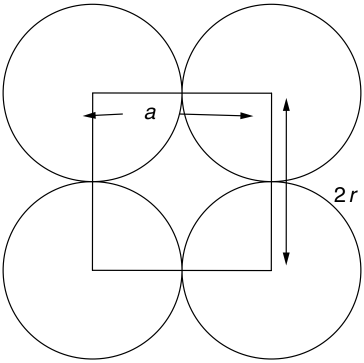
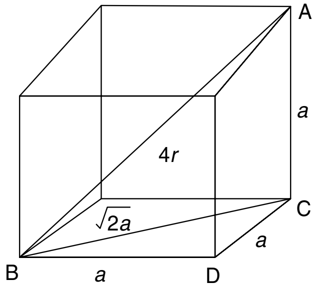


Figure 8 Atomic radius in SCC.

2. For body-centered cubic structure, atoms touch each other along the diagonal of the cube as shown in Fig. 9. Therefore

$$(4\,r)^2 = 3\,(a)^2 \Rightarrow r^2 = 3a^2/16 \Rightarrow r = \sqrt{3}\,a/4$$



$$[(AB)^2 = (BC)^2 - (CA)^2]$$

Figure 9 Atomic radius in BCC.

3. For face-centered cubic structure, atoms touch each other along the diagonal of the any face of the cube as shown in Fig. 10. Therefore

$$(4\,r)^2 = (a)^2 + (a)^2 \Rightarrow r^2 = 2a^2/16 \Rightarrow r = \sqrt{2}\,a/4$$

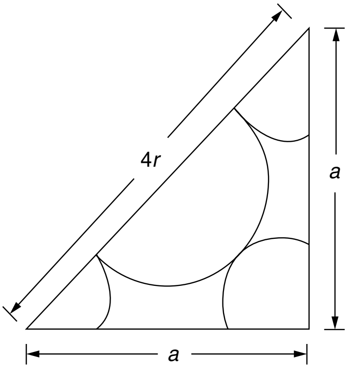


Figure 10 Atomic radius in FCC.

8.10 Co-Ordination Number and Nearest Neighbor Distance

Co-ordination number is defined as the number of nearest neighbors around any point or atom in the crystalline lattice. Generally, $(r + r) = 2r$ or $r_1 + r_2$ when the atoms are in contact with each other is called nearest neighbor distance.

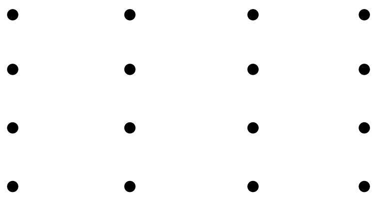
8.11 Crystal Structure

As we have already discussed in Section 8.2 that lattice is a regular periodic arrangement of points in space that look like a net structure and *the points where the atoms are situated in the lattice are known as lattice points. In the simple crystals, there is a single atom or ion at each lattice point. However, in case of compounds, there is a group of two or more atoms or ions at each lattice point. This group of atoms is called a basis. Basis is repeated in space to form the crystal.* Thus, we can obtain crystal structure when a basis is added to each lattice point in space. The logical relation is

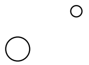
Lattice + Basis = Crystal structure

In other words, one can say that *it is an assembly of one or more atoms which when attached identically to each lattice point gives rise to crystal structure which is reflected through Fig. 11.*
On the basis of crystal structure, crystals may be divided into four classes:

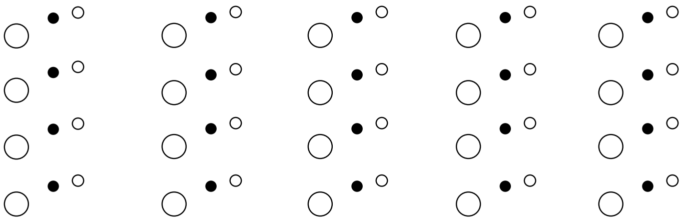
1. Metal crystals.
2. Ionic crystals.
3. Valence crystals.
4. Van der Waals crystals.



(a) Space lattice



(b) Basis containing two different ions



(c) Crystal structure

Figure 11 Formation of crystal structure.

8.11.1 Metal Crystals

Cubic and hexagonal close packed structures are known as metallic crystals. In cubic structure, we find simple cubic (SC), body-centered cubic (BCC) and face-centered cubic (FCC).

8.11.1.1 Simple Cubic Structure (SC)

In this structure, there is one atom at each of the 8 corners of unit cell and each atom is surrounded by six nearest neighbors. Therefore, the co-ordination number is 6. The contribution of each atom to our cell is only $1/8$. So the total number of atom per unit cell is $(1/8)\times 8 = 1$. If a is the distance between the centers of two nearest neighbors, then the atomic radius r is $a/2$ (because atomic radius is equal to half the nearest neighbors distance) as shown in Fig. 12. Let v be the volume of the atoms in the unit cell and V be the volume of the unit cell. The packing factor (PF) is

$$\text{PF} = \frac{v}{V} = \frac{\text{Number of atoms per unit cell} \times \text{Volume of one atom}}{\text{Volume of primitive cell}}$$

$$= \frac{1\times \frac{4}{3}\pi \times \left(\frac{a}{2}\right)^3}{a^3} = \frac{\pi}{6} = 0.52 \text{ or } 52\,\% \qquad \left(\because \text{Volume of one atom} = \frac{4}{3}\pi\, r^3\right)$$

Since the value is low compared to other structures so it is called loosely packed structure. Polonium is an example of this structure.

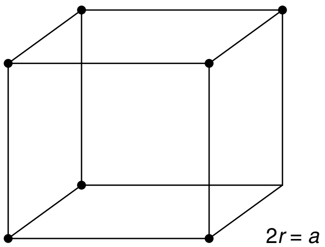


Figure 12 Unit cell of an SCC.

8.11.1.2 Body-Centered Cubic Structure (BCC)

In this structure, there is one atom at each of the eight corners of unit cell and also one atom at the center of the cell. Hence, the unit cell contains $(1/8)\times 8+1=2$ atoms per unit cell. Each atom is surrounded by eight nearest neighbors. Therefore, the co-ordination number is 8. Since, the nearest neighbors distance is $a\sqrt{3}/2$ so that the atomic radius is $a\sqrt{3}/4$ as shown in Fig. 13. The packing factor (PF) is

$$\mathrm{PF}=\frac{v}{V}=\frac{\text{Number of atom per unit cell }\times\text{ Volume of one atom}}{\text{Volume of primitive cell}}$$

$$=\frac{2\times\frac{4}{3}\pi\times\left(\frac{\sqrt{3}a}{4}\right)^3}{a^3}=\frac{\sqrt{3}\,\pi}{8}=0.68\ \text{ or }\ 68\,\%$$

This is a close-packed structure. Cr, Mn, Fe, Ba, etc. are examples of BCC lattices.

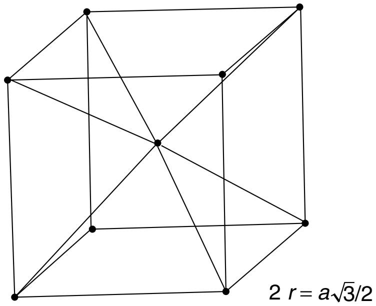


Figure 13 Unit cell of a BCC.

8.11.1.3 Face-Centered Cubic Structure (FCC)

In this structure, there is one atom at each of the eight corners of unit cell and also one atom at the center of the six faces of the cube. Hence, the unit cell contains $(1/8)\times 8+(1/2)\times 6=4$ atoms per unit cell. Each face-centered atom touches the four corner atoms and four face-centered atoms in its own cell and another four face-centered atoms in the outer cell. Therefore, the co-ordination number is 12.

Since, the nearest neighbors distance is $a/\sqrt{2}$ so that the atomic radius is $a/2\sqrt{2}$ (Fig. 14). The packing factor (PF) is

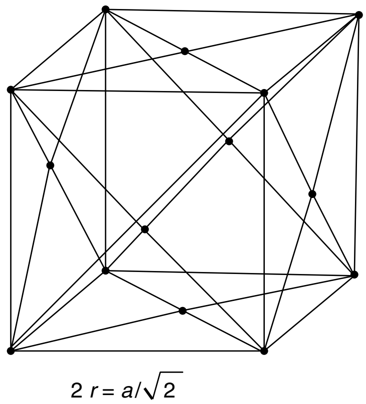


Figure 14 Unit cell of an FCC.

$$\mathrm{PF} = \frac{v}{V} = \frac{\text{Number of atom per unit cell} \times \text{Volume of one atom}}{\text{Volume of primitive cell}}$$

$$= \frac{2 \times \frac{4}{3}\pi \times \left(\frac{a}{2\sqrt{2}}\right)^3}{a^3} = \frac{\pi}{3\sqrt{2}} = 0.74 \text{ or } 74\ \%$$

This structure is highly close-packed compared to SCC and BCC structures. Cu, Ag, Ca, Al are examples of this structure.

8.11.1.4 Hexagonal Close-Pack Structure

It is also closest packing structure with 74% as the packing factor, equal to that of the FCC structure. Number of atoms in an hcp unit cell is obtained through following relation:

$$N = \frac{N_c}{6} + \frac{N_f}{2} + \frac{N_i}{1}$$

where N_c is the total number of corner atoms in the unit cell. There is one atom at each of the 12 corners of the hexagon. N_f is the total number of facial atoms in the cell. There is one atom at the center of the top and bottom faces. Each atom at the center of the top or bottom faces is shared by two hexagonal cells. N_i is the interior atom well within the cell. There are 3 atoms within the cell as shown in Fig. 15.
Hence, the total number of atoms per unit cell is

$$N = \frac{12}{6} + \frac{2}{2} + \frac{3}{1} = 6$$

Since in hcp the corner atoms touch along the edges, each atom touches six atoms in its plane, three atoms below its plane and three atoms above its plane. Therefore, coordination number is 12. Zn, Cd, Mg, Co are some examples of hcp structure.

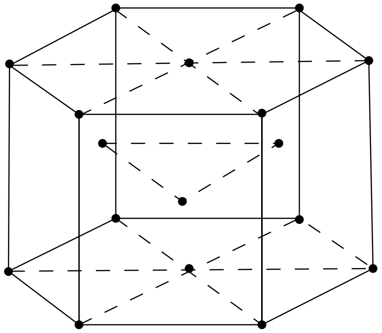


Figure 15 Hexagonal crystal structure.

8.11.2 Ionic Crystal

Basically, ionized crystals come in this group. NaCl, KCl, etc. are some examples of ionic crystals.

8.11.2.1 NaCl Structure

In sodium chloride crystal, there are two different atoms Na and Cl having atomic number 11 and 17, respectively. Sodium atom loses its outermost orbit electron and so acquires an excess of positive charge. On the other hand, chlorine atom absorbs the electron lost by the sodium atom and at the same time acquires an excess of negative charge. In Fig. 16, black dots represent the positions of chlorine ions and the circles that of sodium ions. The unit cell is FCC structure; the basis consists of one Na atom and one Cl atom separated by one-half of the body diagonal of a unit cube. In each unit cube, there are four units of NaCl in which the ions are at the following positions:

Na	$0\quad 0\quad 0$,	$\frac{1}{2}\quad \frac{1}{2}\quad 0$,	$\frac{1}{2}\quad 0\quad \frac{1}{2}$,	$0\quad \frac{1}{2}\quad \frac{1}{2}$
Cl	$\frac{1}{2}\quad \frac{1}{2}\quad \frac{1}{2}$,	$0\quad 0\quad \frac{1}{2}$,	$0\quad \frac{1}{2}\quad 0$,	$\frac{1}{2}\quad 0\quad 0$

Each atom has six nearest neighbors of opposite kind and therefore, the coordination number is 6.

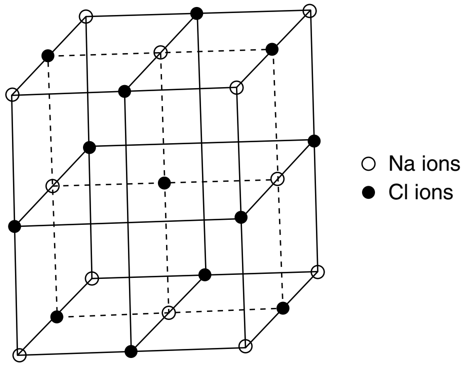


Figure 16 Crystal structure of NaCl.

11.8.3 Valence Crystals

Valence crystals can be obtained by the sharing of valence electrons between neighboring atoms. Diamond, zinc blend, etc. are some examples of the valence crystals.

11.8.3.1 Diamond Structure

Diamond structure is the combination of two interpenetrating face-centered cubic sub-lattices. This structure is shown through Fig. 17. From the figure we see that the points 0 and $1/2$ are on one sublattice; however, the points at $1/4$ and $3/4$ are on the other sublattice, displaced along the body of the diagonal by one-fourth of its length. Its unit cell has in all eight atoms with a basis of two atoms located at 0 0 0, $\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}$.

Its structure is loosely packed structure. The coordination number is four and consequently the atomic packing factor is very low at 34%. The number of atoms per unit cell of diamond structure is 8. If *a* is the distance between the centers of two nearest neighbors and *r* is the radius of each atom, then the relation between *a* and *r* is

$$a = \frac{8r}{\sqrt{3}}$$

C, Si, grey tin are examples of this structure.

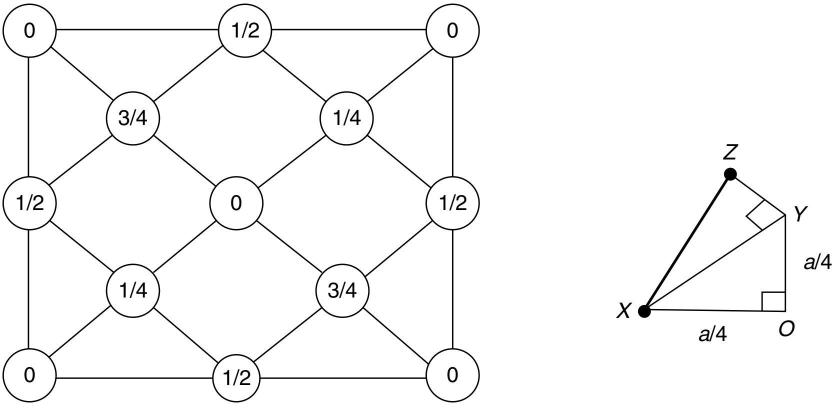


Figure 17 Crystal structure of diamond.

11.8.4 van der Waals Crystals

The rare gases such as argon and organic compounds such as tetraphenyl methane have van der Waals crystal structure.

8.12 Lattice Planes and Miller Indices

The parallel equidistant planes in a crystal which contain a large number of atoms are known as lattice planes and are described by three integers *h*, *k*, *l* simply enclosed in parentheses. *Thus, Miller indices are nothing but the notations used to describe lattice planes and directions in the crystal.* In other words, the reciprocals of the intercepts made by a plane on the crystallographic axes when reduced to smallest numbers are known as Miller indices. Parallel planes may have the same Miller indices.

Following are the rules to determine the Miller indices:

1. Find the intercepts on the axes in terms of the lattice constants *a*, b and c. For example, consider 2*a*, 3*b* and 4*c* as the intercepts of a plane.
2. Express the intercepts as multiplies of lattice parameters along the respective axes. For the plane, these are 2*a/a*, 3*b*/*b* and 4*c*/*c* that is 2, 3 and 4.
3. Take the reciprocal of these numbers, that is, 1/2, 1/3 and 1/4.
4. Reduce these fractions to the smallest triad of integers *h*, *k*, *l* having the same ratio. The quantity (*h k l*) is then the Miller index of that system of planes.

8.12.1 Interplaner Spacing in Terms of Miller Indices

Consider a family of parallel planes (*h k l*). The spacing *d* between the adjacent planes of this family can be calculated by taking any lattice point as origin, erecting coordinate axes in ***a, b, c*** directions and finding the perpendicular distance between this origin and the plane which is nearest to this origin. This plane would obviously have *a/h, b/k* and *c/l* as intercepts on the axes ***a, b, c***, respectively. Now

$$\frac{a}{h}\cos\alpha = d \;\Rightarrow\; \cos\alpha = \frac{d}{a/h}$$

$$\frac{b}{k}\cos\beta = d \;\Rightarrow\; \cos\beta = \frac{d}{b/k}$$

$$\frac{c}{l}\cos\gamma = d \;\Rightarrow\; \cos\gamma = \frac{d}{c/l}$$

where *d* is the interplanar spacing. But we know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Substituting the values of $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ in this above equation we get

$$\left(\frac{dh}{a}\right)^2 + \left(\frac{dk}{b}\right)^2 + \left(\frac{dl}{c}\right)^2 = 1$$

$$\Rightarrow d^2\left\{\left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 + \left(\frac{l}{c}\right)^2\right\} = 1$$

$$\Rightarrow d = \frac{1}{\sqrt{\left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 + \left(\frac{l}{c}\right)^2}}$$

If $a = b = c$ (for cubic structure) then

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

8.13 Reciprocal Lattices

Up to now we have discussed that in a crystal there exist a number of sets of parallel planes with different orientations and spacing, which can diffract a given beam of X-rays. Practically the visualization of such planes is too difficult. These problems can be simplified by consideration of reciprocal lattice.
To understand the visualization, we adopt the following procedure:

1. As the orientation of a plane is determined by its normal, we draw the normal to various sets of planes.
2. The lengths of the normals are taken proportional to the reciprocals of the interplanar spacing, that is $1/d_{hkl}$.

Thus, we get a group of space lattices. This new arrangement of space lattices in reciprocal space is known as the reciprocal lattice. Reciprocal lattice vector is

$$\vec{\sigma}_{hkl} = \frac{1}{d_{hkl}}$$

where $1/d_{hkl}$ is the magnitude of reciprocal vector $\vec{\sigma}_{hkl}$ and whose direction is parallel to the normal to the ($h\ k\ l$) plane. Since the normal to a plane can be represented by a unit vector $\hat{n}$, we have

$$\vec{\sigma}_{hkl} = \frac{1}{d_{hkl}}\,\hat{n}$$

For (100) plane

$$\vec{\sigma}_{100} = \frac{1}{d_{100}}\,\hat{n}$$

But we know that

$$\vec{\sigma}_{100} = \frac{1}{d_{100}}\hat{\mathbf{n}} = \frac{\mathbf{b}\times\mathbf{c}}{V} \quad \text{and} \quad V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

Therefore

$$\vec{\sigma}_{100} = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

Similarly, we can write reciprocal lattice vector for the planes (010) and (001) as

$$\vec{\sigma}_{010} = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})} \quad \text{and} \quad \vec{\sigma}_{001} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

In terms of reciprocal translational vector

$$\mathbf{a}^* = \vec{\sigma}_{100} = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

$$\mathbf{b}^* = \vec{\sigma}_{010} = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

$$\mathbf{c}^* = \vec{\sigma}_{001} = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}$$

8.14 Diffraction of X-Rays by Crystal

Diffraction is nothing but the deviation of light from a straight line when an obstacle or an aperture of size comparable to the wavelength of light falls in its path. Diffraction techniques are generally used for the determination of crystal structure. Since wavelength of X-rays is very small (1 Å – 100 Å), therefore, it is not feasible to construct a grating having grating element of the order of 1 Å. In this sense, Laue in 1913 suggested that crystal can be used for the diffraction of X-rays due to periodic arrangement of atoms with spacing of the order of 1 Å.

8.14.1 Laue's Experiment

In Laue's experiment X-rays were allowed to pass through a pinhole in lead screens and therefore, the output is in the form of narrow beam of X-rays. This collimated beam is further allowed to pass through a thin crystal of either zinc sulphide or sodium chloride (Fig. 18). The rays emerging from the crystal form diffraction pattern on the screen. The diffraction pattern consists of a central spot surrounded by many other low-intensity spots spaced in definite manner. The pattern is obtained due to diffraction of X-rays from the various lattice planes. These spots are known as Laue's spots.

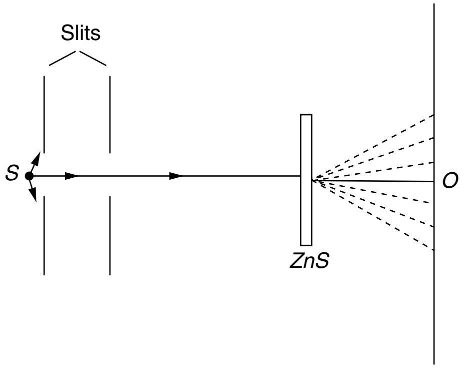


Figure 18 Laue pattern.

The conclusions from Laue's experiment are as follows:

1. The X-rays are electromagnetic radiations of very short wavelength.
2. The atoms in a crystal are arranged in a regular three-dimensional array.

A grating having grating element of the order of wavelength of visible light (4000–7000 Å) is used to study diffraction pattern.

8.14.2 Bragg's Law

In 1912, W. L. Bragg developed a technique to explain the diffraction of X-rays by a crystal. According to Bragg, X-rays can be diffracted by various parallel atomic planes of crystals when the rays are incident on their surface at glancing angle θ. The diffracted beams are found due to constructive interference between refraction from the various planes of atoms.

In order to derive an expression for Bragg's law, consider a set of parallel planes in a crystal with inter-planar spacing d. Suppose an X-ray falls on atoms A and B of atomic planes at a glancing angle θ as shown in Fig. 19.

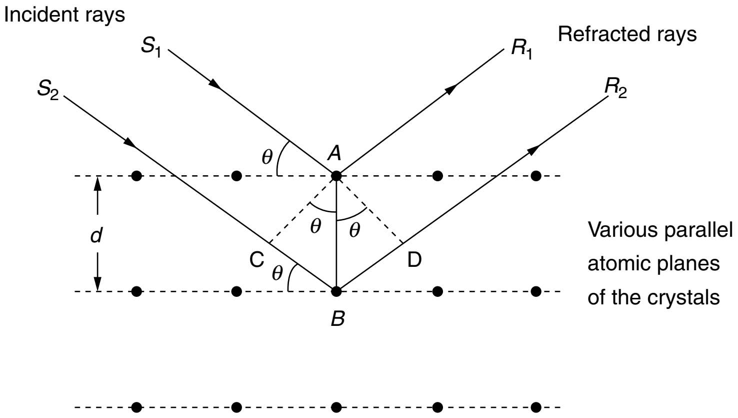


Figure 19 Bragg's Law.

These rays are scattered by atoms A and B along the directions AR_1 and BR_2, respectively. Draw the perpendiculars AC and AD on S_1A and S_2B. The path difference between the two rays is

$$\Delta = CB + BD$$

From Δ ACB,

$$\sin\theta = CB/AB \Rightarrow CB = AB \sin\theta = d\sin\theta$$

Similarly, from Δ ADB,

$$DB = d\sin\theta$$

Therefore the path difference is

$$\Delta = d\sin\theta + d\sin\theta = 2\,d\sin\theta$$

But we know that constructive interference takes place between the rays reflected from the various atomic planes. Therefore

$$\Delta = \pm\, n\lambda \Rightarrow 2\,d\sin\theta = n\lambda \quad \text{where } n = 1, 2, 3, \ldots$$

This is known as Bragg's law and $n = 1, 2, 3, \ldots$ correspond to first-, second-, third-order spectrum and so on.

8.14.3 Bragg's X-Rays Spectrometer

For the detailed study of crystal structure, Bragg and his son designed an apparatus known as Bragg's spectrometer. In this apparatus there is a turn table which is capable of rotation about a vertical axis passing through its center. The rotation can be read on a circular graduated scale V_1. X-rays from an X-ray tube are made incident through slits S_1 and S_2 to obtain a narrow beam, which is then allowed to fall on a crystal C mounted on turn table. The reflected beam enters the ionization chamber (which is used to measure the intensities of the diffracted X-rays in terms of ionization current) filled with ethyl bromide. The position of the ionization chamber can be read through V_2 as shown in Fig. 20.

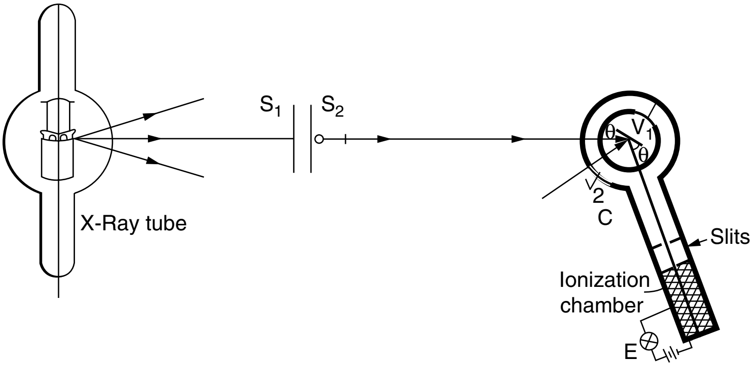


Figure 20 Bragg's spectrometer.

The angle for which the diffraction intensities are maximum gives the value of θ which satisfies the Bragg's law

$$2d \sin \theta = n\lambda$$

where $n = 1, 2, 3, \ldots$ corresponds to first-, second-, third-order spectrum and so on.
For first-order spectrum

$$n = 1 \quad \text{and} \quad 2d \sin \theta_1 = \lambda$$

For second-order spectrum

$$n = 2 \quad \text{and} \quad 2d \sin \theta_2 = 2\lambda$$

For third-order spectrum

$$n = 3 \quad \text{and} \quad 2d \sin \theta_3 = 3\lambda$$

Hence

$$\sin \theta_1 : \sin \theta_2 : \sin \theta_3 = 1 : 2 : 3$$

Thus, the glancing angle can be directly measured from the graph for the different orders. If a graph is plotted between the ionization current and the glancing angles, some peaks A_1, A_2 and A_3 are obtained, which represent the values of glancing angles satisfying Bragg's law as shown in Fig. 21(a). If the two wavelengths are used then corresponding B_1, B_2 and B_3 peaks are also observed satisfying Bragg's law [Fig. 21(b)].

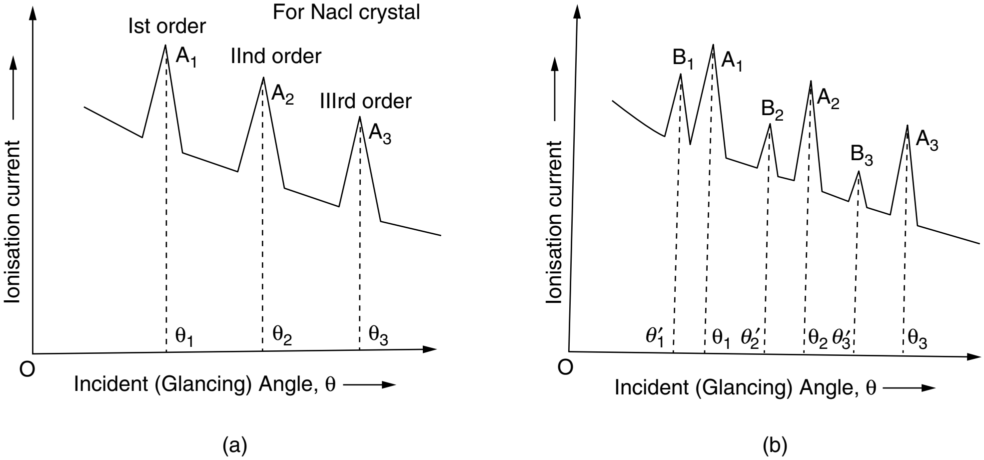


Figure 21 The glancing angle for different orders and wavelength λ_1 and λ_2.

Solved Examples

Example 1

Calculate the lattice parameters of NaCl crystal if it's density (ρ) is 2189 Kgm^{-3}and the Avogadro constant N_A is 6.02×10^{23}/ gm-mole.

Solution: Sodium chloride is a cubic crystal and it's lattice is FCC. The lattice constant a is

$$a = \left(\frac{nM}{N\rho}\right)^{1/3}$$

where M is the molecular weight of NaCl, ρ is the density of NaCl crystal, $n = 4$ (NaCl has FCC lattice). So

$$a = \left[\frac{4 \times 58.5}{6.02 \times 10^{26} \times 2189}\right]^{1/3} = (177 \times 10^{-30}\,\mathrm{m}^3)^{1/3} = 5.61 \times 10^{-10}\,\mathrm{m} = 5.61\ \text{Å}$$

Example 2

Copper has a density of 8.96 gm/cm^3 and an atomic weight of 63.5. Calculate the distance between the two nearest copper atoms in the FCC structure. Given, Avogadro constant is 6.02 × 10^{23}/ gm-mole.

Solution: The lattice constant a is

$$a = \left(\frac{nM}{N\rho}\right)^{1/3} = 3.61 \times 10^{-8}\,\mathrm{cm} = 3.61\ \text{Å}$$

The nearest neighbour distance in the FCC lattice is $a/\sqrt{2}$. Therefore the distance between two nearest neighbour Cu atoms is

$$d = \frac{a}{\sqrt{2}} = \frac{3.61}{\sqrt{2}}\ \text{Å} = 2.55\ \text{Å}$$

Example 3

The density of α-iron is 7870 Kgm $^{-3}$ and its atomic weight is 55.8. If it crystalizes in FCC lattice, deduce its lattice constant. The Avogadro constant N_A is 6.02 × 10^{23}/ gm-mole.

Solution: For an FCC lattice, $n = 4$. Therefore

$$a = \left(\frac{nM}{N_A \times 100 \times \rho}\right)^{1/3} = \left[\frac{2 \times 55.8}{6.02 \times 10^{23} \times 7870}\right]^{1/3}$$

$$= [23.5 \times 10^{-30}]^{1/3} = 2.86 \times 10^{-10}\ \mathrm{m} = 2.86\ \text{Å}$$

Example 4

In a crystal, a lattice plane cuts intercepts of a, $2b$ and $3c$ along the three axes where $\vec{a}, \vec{b}, \vec{c}$ are the primitive vectors of the unit cell. Determine the Miller indices of the given plane.

Solution: From the law of rational indices, we have

$$a : 2b : 3c = \frac{a}{h} : \frac{b}{k} : \frac{c}{l}$$

where h,k,l are the Miller indices. Therefore

$$\frac{1}{h} : \frac{1}{k} : \frac{1}{l} = 1 : 2 : 3 \Rightarrow h : k : l = 1 : \frac{1}{2} : \frac{1}{3} = 6 : 3 : 2$$

Therefore, the plane is (6 3 2).

Example 5

Deduce the Miller indices of a plane in an orthorhombic crystal which cuts intercepts of $3a$, $-2b$, $3c/2$ along the three axes.

Solution: From the law of rational indices, we have

$$3a : -2b : \frac{3c}{2} = \frac{a}{h} : \frac{b}{k} : \frac{c}{l}$$

$$\Rightarrow 3 : -2 : \frac{3}{2} = \frac{1}{h} : \frac{1}{k} : \frac{1}{l}$$

$$\Rightarrow \frac{1}{3} : -\frac{1}{2} : \frac{2}{3} = h : k : l$$

$$\Rightarrow (2 : -3 : 4) = h : k : l$$

Therefore Miller indices = $(2, \bar{3}, 4)$.

Example 6

Calculate the Miller indices of a plane which intercepts the three axes in ratio $a : 3b : 4c$, where a, b, c are primitive lattice translation vectors.

Solution: Comparing the ratios with intercepts made by a $(h\ k\ l)$ plane we get

$$a : 3b : 4c :: \frac{a}{h} : \frac{b}{k} : \frac{c}{l}$$

or $$1:3:4::\frac{1}{h}:\frac{1}{k}:\frac{1}{l}$$

or $$h:\ k:l::1:\frac{1}{3}:\frac{1}{4}$$

or $$h:\ k:l::12:4:3$$

Hence, the Miller indices (*h k l*) for the plane are (12, 4, 3).

Example 7

Determine the interplanar spacing of a lattice plane in a simple cubic lattice with edge 2 Å which cuts the axes in intercepts ratio 3 : 4 : 5.

Solution: According to the relation between interplanea spacing and Miller indices

$$d=\frac{1}{\sqrt{\left(\frac{h}{a}\right)^2+\left(\frac{k}{b}\right)^2+\left(\frac{l}{c}\right)^2}}$$

If $a = b = c = 2$ (for cubic structure) then

$$d=\frac{a}{\sqrt{h^2+k^2+l^2}}$$

We have

$$3:4:5::\frac{a}{h}:\frac{b}{k}:\frac{c}{l}$$

or $$3:4:5::\frac{2}{h}:\frac{2}{k}:\frac{2}{l}$$

or $$h:\ k:l::\frac{2}{3}:\frac{2}{4}:\frac{2}{5}$$

or $$h:\ k:l::40:30:24$$

So

$$d=\frac{2}{\sqrt{40^2+30^2+24^2}}=\frac{2}{\sqrt{2976}}\text{ Å}$$

Example 8

Calculate the glancing angle of the (1 1 0) plane of a simple cubic crystal ($a = 2.814$ Å) corresponding to the second-order diffraction maximum for X-rays of wavelength 0.710 Å.

Solution: The distance between successive lattice planes is given by

$$d_{hkl} = \frac{a}{\sqrt{h^2 + k^2 + l^2}}$$

$$\Rightarrow d_{110} = \frac{2.814}{\sqrt{1^2 + 1^2 + 0^2}} = \frac{2.814}{\sqrt{2}} \text{ Å}$$

From the Bragg equation for X-ray diffraction

$$2d_{110} \sin\theta = n\lambda$$

Therefore

$$\sin\theta = \frac{n\lambda}{2d_{110}} = \frac{2\times 0.710}{2\times 1.990} = 0.357 \Rightarrow \theta = \sin^{-1}(0.357) = 21°$$

Example 9

Electrons are accelerated by 844 V and are reflected from a crystal. The reflection maximum occurs when the glancing angle is 58°. Determine the spacing of the crystal.

Solution: The wavelength of the wave associated with electrons is

$$\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mE}} \left(\because E = \frac{1}{2}mv^2\right) = \frac{h}{\sqrt{2meV}} \quad (\because E = eV)$$

We have m = the mass of the election = 9.1×10^{-31} kg, e = the charge of electron = 1.6×10^{-19} C, V = accelerating potential = 844 V and h = Planck constant = 6.626×10^{-34} Js. Using these values, $\lambda = 0.042$ nm. From Bragg's law,

$$2d\sin\theta = n\,\lambda$$

$$\Rightarrow d = \frac{\lambda}{2\sin\theta} \qquad (n = 1 \text{ for reflection maximum })$$

$$= \frac{0.042}{2\times\sin 58°} = 0.025 \text{ nm}$$

Short Answers of Some Important Questions

1. Distinguish between amorphous and crystalline solids.

   Answer: In crystalline solids, the atoms or molecules are arranged in regular repeated pattern. So, these materials have long-range orders in their structures. However, the amorphous solids are those which lack the regular arrangement of atoms or molecules. So, these materials have short-range orders in their structure.

2. What do you mean by primitive cell?

   Answer: The smallest volume that contains the full pattern is called a unit cell. If a unit cell contains lattice points only at its corners then it is called a primitive unit cell or simple unit cell.

3. What is lattice?

   Answer: The periodic arrangement of atoms in a crystal is called a lattice.

4. Explain plane lattice and space lattice.

   Answer: An array of points where the environment around any one point is identical with that of any other point is called plane lattice. If plane lattice is studied in three dimensions then it is said to be space lattice.

5. Explain space lattice in terms of lattice points.

   Answer: The points where the atoms are situated in the lattice are known as lattice points and the space with these points is known as space lattice.

6. Distinguish between primitive cell and unit cell.

   Answer: A unit cell which has only one lattice point per unit volume is known as a primitive cell. The remaining cells are called unit cells. All primitive cells are unit cells but all unit cells may not be primitive cells.

7. How many primitive cells are there in Bravais lattices?

   Answer: There are only seven primitive cells in Bravais lattices.

8. How is a crystal structure obtained?

   Answer: A crystal structure is obtained by combining the space lattice with a basis.

9. What are Miller indices?

   Answer: The reciprocals of the intercepts made by a plane on the crystallographic axes when reduced to smallest numbers are known as Miller indices. Parallel planes may have the same Miller indices.

10. What is reciprocal lattice?

    Answer: The arrangement of space lattices in reciprocal space is known as the reciprocal lattice.

11. Explain diffraction of X-rays.

    Answer: Since wavelength of X-rays is very small (1–100 Å), therefore, it is not feasible to construct a grating having grating element of the order of 1 Å. In this sense, Laue in 1913 suggested that crystal can be used for the diffraction of X-rays due to periodic arrangement of atoms with spacing of the order of 1 Å.

Important Points and Formulas

1. A crystal is a solid in which it is possible to choose a small group of atoms or molecules which can be assumed to be considered in a regular-sided 'box' or unit cell.
2. Lattice is a regular periodic arrangement of points in space that look like a net structure.
3. The points where the atoms are situated in the lattice are known as lattice points and the space with these points is known as space lattice.
4. The parallelepiped formed by using the basis vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ as concurrent edges is called the unit cell of the space lattice.

5. There are 14 possible ways to arrange atoms in a unit cell under the seven crystal systems. These are frequently called the Bravais lattices.
6. Atomic radius is half the distance between the nearest neighbors.
7. The total number of atom per unit cell in SCC is $(1/8) \times 8 = 1$.
8. The total number of atom per unit cell in BCC is $(1/8) \times 8 + 1 = 2$
9. The total number of atom per unit cell in FCC is $(1/8) \times 8 + (1/2) \times 6 = 4$
10. The reciprocals of the intercepts made by a plane on the crystallographic axes when reduced to smallest numbers are known as Miller indices. Parallel planes may have the same Miller indices.
11. New arrangement of space lattices in reciprocal space is known as the reciprocal lattice. Reciprocal lattice vector is

$$\vec{\sigma}_{hkl} = \frac{1}{d_{hkl}}$$

12. $2d \sin\theta = n\lambda$, where $n = 1, 2, 3, \ldots$. This is known as Bragg's law and $n = 1, 2, 3, \ldots$ correspond to first-, second-, third-order spectrum and so on.

Multiple Choice Questions

1. A regular arrangement of atoms in a matter is known as
   (a) crystal **(b)** amorphous
   (c) materials **(d)** None of the above

2. The individual crystals have dimension
   (a) 10^{-7} cm **(b)** 10^{-4} cm
   (c) 10^{-10} m **(d)** 10^{-10} cm

3. The crystal structure can be studied with
   (a) 1 – 100 Å **(b)** 4000 – 6000 Å
   (c) Both (a) and (b) **(d)** None of the above

4. The wavelength of X-rays is
   (a) 10^{-8} m **(b)** 10^{-6} cm
   (c) 10^{-3} m **(d)** 10^{-10} m

5. The array of lattice point in three-dimensional crystal is called
   (a) basis **(b)** space lattice
   (c) crystal structure **(d)** unit cell

6. Crystal formation consists of
   (a) basis and unit cell
   (b) unit and lattice point
   (c) basis and lattice point
   (d) None of the above

7. Basis of the crystal is the
   (a) arrangement and orientation of atoms
   (b) regular arrangement of atoms
   (c) arrangement of atoms
   (d) orientation of atoms

8. A crystalline substance has basis containing more than two atoms. Such basis is called
   (a) triatomic **(b)** diatomic
   (c) multiatomic **(d)** tetra-atomic

9. If $\alpha = \beta = 90 \neq \gamma$ and $a \neq b \neq c$ then crystal system is
   (a) cubic crystal system
   (b) monoclinic
   (c) triclinic
   (d) tetraclinic

10. Which of the following is correct for an orthorhombic crystal system?
    (a) $\alpha = \beta = \gamma = 90$ and $a \neq b \neq c$
    (b) $\alpha = \beta = \gamma = 90$ and $a \neq b = c$
    (c) $\alpha = \beta = 90,\ \gamma = 120$ and $a \neq b \neq c$
    (d) $\alpha = \beta = 90,\ \gamma \neq 90$ and $a \neq b \neq c$

11. The coordination number in case of simple cubic crystal lattice is
    (a) 12 **(b)** 2
    (c) 6 **(d)** 1

12. In a simple cubic lattice the ratio $d_{100} : d_{110} : d_{111}$ is
    (a) 6 : 3 : 1 **(b)** $\sqrt{3} : \sqrt{6} : 1$
    (c) $6 : 3 : \sqrt{2}$ **(d)** $\sqrt{6} : \sqrt{3} : \sqrt{2}$

13. Which of the following Bravais lattice is not found in a cubic crystal?
(a) Simple cubic **(b)** Face centered
(c) Body centered **(d)** Base centered

14. The packing fraction of a diamond crystal structure is
(a) 34% **(b)** 52%
(c) 68% **(d)** 74%

15. Which one of the following unit cell is a primitive cell?
(a) Simple and cubic **(b)** BCC
(c) FCC **(d)** None of the above

16. The packing factor of FCC structure is
(a) 52% **(b)** 68%
(c) 92% **(d)** None

17. The number of atoms per unit cell in BCC is
(a) 1 **(b)** 2
(c) 4 **(d)** 8

18. The number of atoms per unit cell in FCC is
(a) 1 **(b)** 2
(c) 4 **(d)** 8

19. The atomic packing fraction for a face centered cubic cell is
(a) 0.34 **(b)** 0.52
(c) 0.68 **(d)** 0.74

20. Relation between atomic radius R and lattice constant a in a body centered cubic cell is
(a) $R = a/2$ **(b)** $R = a\sqrt{3}/4$
(c) $R = a\sqrt{3}/2$ **(d)** $R = a/2\sqrt{2}$

Short Answer Type Questions

1. What is a crystalline solid?
2. Define crystal structure.
3. What are crystal and Bravais lattices?
4. What is primitive cell?
5. What are plane lattice and space lattice?
6. Define Miller indices.
7. Explain reciprocal lattice.
8. What do you understand by Laue's method?
9. Explain Bragg's law.
10. What do you mean diffraction of X-rays?
11. Write the conclusion of Laue's method.

Long Answer Type Questions

1. Explain crystalline and amorphous solids.
2. Define crystal structure, crystal lattice and Bravais lattice. Also describe how atoms are arranged in cubic crystal.
3. What do you understand by crystal systems? Explain 14 types of space lattices.
4. Explain simple cubic, base-centered cubic structure and face-centered cubic structures.
5. What is crystal structure? Explain its types.
6. Describe hexagonal close pack structure.
7. What are Miller indices? How are they obtained?
8. What do you mean by reciprocal lattice?
9. Explain diffraction of X-rays by Laue's crystal diffraction.
10. What are Laue spots? Explain diffraction of X-rays by Laue experiment.
11. What is Bragg's law? Explain how the wavelength of given X-rays beam is determined.
12. Describe Bragg's spectrometer and explain how it is used to study the crystal structure.
13. Derive Bragg's equation for reflection of X-ray by crystal planes.

Numerical Problems

1. Calculate Miller indices of a plane which intercepts the three axes in ratio $2a : 4b : 1c$, where a, b, c are primitive lattice translation vectors.

2. Determine interplanar spacing of a lattice plane in a simple cubic lattice with edge 1 Å which cuts the axes in intercepts ratio 2:3:4.

3. In Bragg's reflection for X-ray a reflection was found at 30° glancing angle with lattice planes of spacing 1.87 Å. If this is second-order reflection, calculate the wavelength of X-rays.

Answers

Multiple Choice Questions

1.	(a)	**5.**	(b)	**9.**	(b)	**13.**	(d)	**17.**	(b)
2.	(b)	**6.**	(c)	**10.**	(a)	**14.**	(a)	**18.**	(c)
3.	(a)	**7.**	(a)	**11.**	(c)	**15.**	(a)	**19.**	(d)
4.	(d)	**8.**	(c)	**12.**	(d)	**16.**	(d)	**20.**	(b)

Numerical Problems

1. (4 2 8)

2. 0.13 Å

3. 0.935 Å

9

Dielectrics

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Dielectric constant and polarization of dielectric materials.
- Types of polarization.
- Polarizability.
- Equation of internal fields in liquid and solid (one-dimensional).
- Clausius–Mossoti equation.
- Frequency dependence of dielectric constant.
- Dielectric losses.
- Important applications of dielectric material.

9.1 Introduction

So far we have restricted ourselves to a discussion of the electric field in free space or on the surface of a conductor. Although we are familiar with dielectrics studied in the earlier classes, but have not discussed how they react in the presence of applied external electric fields. In this chapter, we shall study the effect of dielectric materials under the action of external electric field. First of all, we have to be familiar with matter. Matter of course comes in many forms – solids, liquids, gases, metals, etc. Basically, dielectrics are non-conducting materials like insulators. So first of all we have to know the difference between dielectric and insulator. There is a slight difference between dielectric and insulator which is the function they perform. The main function of a dielectric material is to store electrical energy in the form of potential energy while the function of an insulator is to obstruct of flow of current.

In dielectric or insulator, positive and negative charges are tightly bound and do not contain any free charge carriers for conduction. If these materials are subjected to an external electric field, positive and negative charges are displaced from their equilibrium position and they form electric dipoles, which are responsible for the characteristic behavior of dielectric materials as shown in Fig. 1. On the basis of the interaction of the field with the dielectric, **linear**, **homogeneous** and **isotropic** are the types of dielectric.

Let us see what happen if dielectric materials are placed between the plates of a parallel plate capacitor. The answer is: The capacitance of the capacitor increases. This is due to the polarization of dielectric materials, where positive charges shifting slightly towards one side and the negative charges towards the other side; this causes the formation of electric dipoles. As a result, an additional field $\vec{E}_1$ is generated due to induced charges of dielectric slab which opposes the applied external electric field $\vec{E}_0$. Thus, the resultant electric field $\vec{E}$ decreases as a result of the capacitor increases:

$$\vec{E} = \vec{E}_0 - \vec{E}_1 \tag{9.1}$$

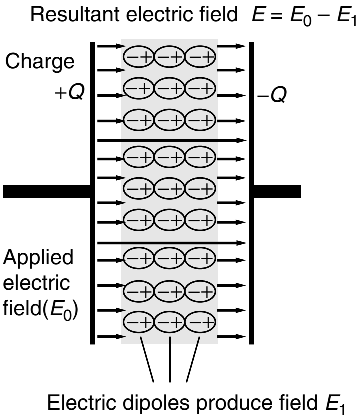


Figure 1 Polarized molecules in presence of external electric field.

9.2 Dielectric Constant

We have studied that capacitance C of a parallel plate capacitor having area A, permittivity ε and distance d between the plates can be expressed as

$$C = \frac{\varepsilon A}{d} \tag{9.2}$$

If the medium is air, then the capacitance C_0 of the same capacitor is given as

$$C_0 = \frac{\varepsilon_0 A}{d} \tag{9.3}$$

where ε_0 is the permittivity of the free space. From Eqs. (9.2) and (9.3) we can write

$$\frac{C}{C_0} = \frac{\varepsilon}{\varepsilon_0} \tag{9.4}$$

But we know that $\varepsilon/\varepsilon_0 = \varepsilon_r$ where ε_r is relative permittivity; it is also known as dielectric constant K.

Thus, dielectric constant is the ratio of the capacitance of the capacitor filled with dielectric material to the capacitance of the capacitor with air medium for the same capacitor.

If we talk about electric field and potential difference then the dielectric constant of the material can be given as

$$K = \varepsilon_r = \frac{C}{C_0} = \frac{\varepsilon}{\varepsilon_0} = \frac{E_0}{E} = \frac{V_0}{V} \tag{9.5}$$

9.3 Polar and Non-Polar Molecules

We have discussed in Section 9.1 that the molecular positive and negative charges in equal magnitudes are displaced from their equilibrium position. The positive charges are concentrated at a single point called center of gravity of the positive charges; and the negative charges are concentrated at other single point called center of gravity of negative charges.

When centers of gravity of positive and negative charges coincide, the molecule is said to be ***non-polar****. The non-polar molecules have zero permanent dipole moment due to symmetrical structure*. Some examples of non-polar molecules are H_2, N_2, Cl_2, O_2,CCl_4, BF_3, CO_2, etc. as shown in Fig. 2.

When the centers of gravity of positive and negative charges do not coincide but are rather separated by some distance, the molecule is said to be ***polar****. The polar molecules have permanent dipole moment due to non-symmetrical structure*. Some examples of polar molecules are H_2O, HCl, NH_3, CH_3Cl, CO, etc. as shown in Fig. 2.

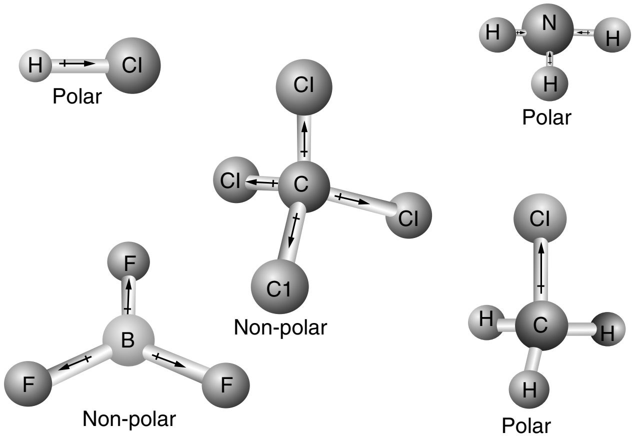


Figure 2 Structure of polar and non-polar molecules.

9.4 Dielectric Polarization

When a dielectric material is subjected to an external electric field then dipoles are created. These dipoles acquire an induced electric dipole moment due to the displacement of charged particles in the direction of the applied external electric field. *This induced dipole moment per unit volume is called polarization and the materials are said to be polarized.*

It has been observed that induced dipole moment $\overline{p}$ is directly proportional to the applied external electric field, that is,

$$\overline{p} = \alpha \cdot \overline{E} \tag{9.6}$$

where α is known as proportionality constant and is said to be atomic or molecular polarizibility. If N is the total number of atoms per unit volume then according to the definition of polarization (P)

$$P = N \cdot p = N\alpha E \tag{9.7}$$

9.5 Types of Polarization

There are four types of polarization based on different atomic mechanisms:

1. Electronic polarization.
2. Ionic polarization.
3. Orientational polarization.
4. Space charge polarization.

These will be discussed in the following subsections.

9.5.1 Electronic Polarization

Electronic polarization occurs due to displacement of electron clouds from the nucleus. Usually in an atom, nucleus and electron clouds are tightly attached and electrons revolve around the nucleus. If the same atom is subjected to an external electric field then center of electrons cloud and positive nucleus are displaced due to Lorentz force till it is equal to a Colombian force developed between them as shown in Fig. 3. Let the center of electrons cloud be displaced with *x* distance from center of positive nucleus.

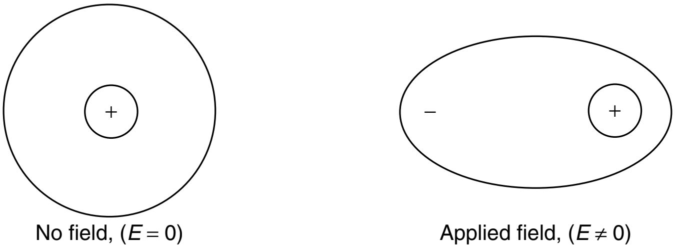


Figure 3 Displacement of electron clouds from nucleus in presence of electric field.

Suppose if the charges on electron clouds and nucleus are –*Ze* and +*Ze*, respectively. Then the volume charge density is

$$\rho = \frac{\text{Charge}}{\text{Volume}} = \frac{-Ze}{(4/3)\pi R^3} \tag{9.8}$$

where R is the radius of the sphere. In the presence of external electric field induced charge due to displacement

$$\text{Charge} = \text{Volume} \times \rho = \frac{4}{3}\pi x^3 \times \frac{-3}{4}\frac{Ze}{\pi R^3} = -Ze\frac{x^3}{R^3} \tag{9.9}$$

Hence, Colombian force due to induced charge is

$$\text{Colombian force} = \frac{1}{4\pi\varepsilon_0}(Ze)\left(-Ze\frac{x^3}{R^3}\right) \tag{9.10}$$

Since Lorentz force must be equal to Colombian force, so

$$-ZeE = \frac{1}{4\pi\varepsilon_0}(Ze)\left(-Ze\frac{x^3}{R^3}\right) \tag{9.11}$$

$$\Rightarrow x = \frac{4\pi\varepsilon_0 R^3 E}{Ze} \tag{9.12}$$

According to the definition of dipole moment

$$p_e = \text{Charge} \times \text{Displacemen} = Ze \times \frac{4\pi\varepsilon_0 R^3 E}{Ze} \tag{9.13}$$

We know that

$$P_e = Np_e = 4\pi\varepsilon_0 R^3 NE \tag{9.14}$$

Thus Eq. (9.14) represents that electronic polarization or polarizing vector is directly proportional to the applied external electric field, that is

$$P_e \propto E \tag{9.15}$$

or

$$P_e = N\alpha_e E \tag{9.16}$$

where N is the total number of atoms per unit volume and α_e is the electronics polarizibility and it is temperature independent.

Now comparing Eqs. (9.14) and (9.15) we get

$$\alpha_e = 4\pi\varepsilon_0 R^3 \tag{9.17}$$

9.5.2 Ionic Polarization

Ionic polarization occurs due to the displacement of ions under the action of external electric field which gives rise to a net dipole moment as shown in Fig. 4. It has been observed that dipole moment of ionic materials due to ionic polarization is directly proportional to the applied external electric field:

$$p_i \propto E \tag{9.18}$$

$$p_i = \alpha_i E \tag{9.19}$$

where α_i is known as ionic polarizibility and it is also temperature independent. If N is the total number of atoms per unit volume, then the ionic polarization is given by

$$P_i = N\alpha_i E \tag{9.20}$$

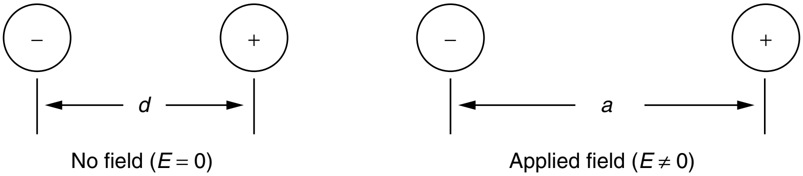


Figure 4 Ionic polarization.

For most of the materials the ionic polarizibility is less than the electronic polarizibility:

$$\alpha_i = \frac{1}{10}\alpha_e \tag{9.21}$$

9.5.3 Orientational Polarization

In absence of an external electric field, atoms or molecules are randomly oriented. Hence, the net dipole moment of the material is zero. However, in presence of external electric field, a torque is exerted on the dipoles and the dipoles tend to align themselves along the direction of applied external electric field and the dielectrics get polarized as shown in Fig. 5. This type of polarization which occurs in polar substances is known as orientational polarization and is represented by P_0.

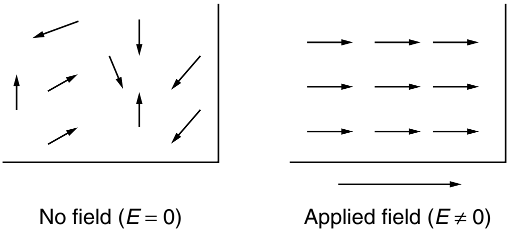


Figure 5 Orientational polarization.

It has been observed that dipole moment of polar molecules due to orientational polarization is directly proportional to the applied external electric field:

$$p_0 \propto E \tag{9.22}$$

$$\Rightarrow p_0 = \alpha_0 E \tag{9.23}$$

where α_0 is known as orientational polarizibility and it is highly temperature dependent; it decreases with increase in temperature. If N is the total number of atoms per unit volume, then the orientational polarization is given by

$$P_0 = N\alpha_0 E \tag{9.24}$$

The orientational polarizability α_0 in terms of Boltzmann's constant k and absolute temperature T is

$$\alpha_0 = \frac{p_0^2}{3kT}$$

9.5.4 Space Charge Polarization

The space charge polarization is nothing but redistribution of charges in the presence of applied external electric field which is not much significant in most of the dielectric materials. It has been observed that dipole moment of dielectric materials due to space charge polarization is directly proportional to the applied external electric field:

$$p_s \propto E \tag{9.25}$$

$$\Rightarrow p_s = \alpha_s E \tag{9.26}$$

where α_s is known as space charge polarizibility and it is also temperature dependent. If N is the total number of atoms per unit volume, then the space charge polarization is given by

$$P_s = N\alpha_s E \tag{9.27}$$

9.6 Displacement Vector

We know that the electric field strength E at a distance r from a charge q is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{q\vec{r}}{\vec{r}^3} \tag{9.28}$$

From this equation we can say that electric field strength $\vec{E}$ depends not only on position $\vec{r}$ and magnitude of the charges but also on dielectric medium permittivity ε in which the field is measured. It is desirable to associate the charge q with an additional second electrical quantity which will be independent of medium involved. This second electrical quantity is known as electric displacement vector and is represented by $\vec{D}$. The unit of displacement vector is coulomb/m^2:

$$\vec{D} = \frac{1}{4\pi}\frac{q\vec{r}}{\vec{r}^3} \tag{9.29}$$

Comparing Eqs. (9.28) and (9.29) we get

$$\vec{D} = \varepsilon\ \vec{E} \tag{9.30a}$$

or

$$\vec{D} = \varepsilon_0\varepsilon_r\ \vec{E} \qquad \text{(for free space } \varepsilon_r = 1) \tag{9.30b}$$

9.7 Relation between *D, E* and *P*

We know that the resultant field inside the dielectric material is

$$\vec{E} = \vec{E}_0 - \vec{E}_1 \tag{9.31}$$

where $\vec{E}$ is the resultant electric field, $\vec{E}_0$ is the applied field and $\vec{E}_1$ is the field create by dielectric material. If σ and σ' are the surface charge density of free and bound (induced) charges on the capacitor plates and dielectric within the plates respectively, then the magnitudes of electric field due to surface charge densities are

$$E_0 = \frac{\sigma}{\varepsilon_0} \quad \text{and} \quad E_1 = \frac{\sigma'}{\varepsilon_0} \tag{9.32}$$

Now putting the value of E_0 and E_1 from Eq. (9.32) in Eq. (9.31) to determine the field E we get

$$E = \frac{\sigma}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0} \quad \text{or} \quad \varepsilon_0 E = \sigma - \sigma' \tag{9.33}$$

Mathematically, $\sigma = q'/A$ and if we multiply and divide by some finite distance, say *d*, then the product of charge and distance is dipole moment and the product of area and distance is volume. Hence, dipole moment per unit volume is known as polarization. So surface charge density σ' is equal to the magnitude of polarization *P*. Thus Eq. (9.33) becomes

$$\sigma = P + \varepsilon_0 E \tag{9.34}$$

From Eq. (9.29) we can say that magnitude of displacement vector *D* represents the surface charge density of free charges. Therefore Eq. (9.34) takes the form

$$D = \varepsilon_0 E + P \tag{9.35}$$

Thus, *D* is related to only free charges, *P* is related to bound charges; however, *E* is related to both free and bound charges.

9.8 Relation between *P* and *K*

Since surface charge density σ' is equal to the magnitude of polarization *P* hence

$$E_1 = \frac{\sigma'}{\varepsilon_0} = \frac{P}{\varepsilon_0} \tag{9.36}$$

We know that

$$\vec{E} = \vec{E}_0 - \vec{E}_1 \Rightarrow E = E_0 - \frac{P}{\varepsilon_0}$$

$$\Rightarrow P = \varepsilon_0 E\left(\frac{E_0}{E} - 1\right)$$

$$\Rightarrow P = \varepsilon_0 E(K - 1) \qquad \text{[using Eq. (9.5)]} \tag{9.37}$$

This is the required expression for polarization vector ***P*** and dielectric constant *K*. This indicates that in free space where $K = 1$, the polarization *P* is zero.

9.9 Relation between Electrical Susceptibility χ_e and K

We know that polarization is directly proportional to the field and is given by the following expression:

$$P = \chi_e E \tag{9.38}$$

Substituting the values of Eqs. (9.30a) and (9.38) in Eq. (9.35) and after solving we get

$$\varepsilon_r = 1 + \frac{\chi_e}{\varepsilon_0} \tag{9.39}$$

This is the required relation.

9.10 Internal Fields in Liquids and Solids

The electric field is very much affected by the presence of external electric field. When a field is applied to a dielectric, their atoms get polarized and become electric dipoles, which also sets up their own electric field. The atoms in the material are closely packed and every atom is surrounded by many other atoms. Therefore, the field at any atomic point is due to the resultant field of all neighboring electric dipoles and the external applied field. This total electric field responsible for producing electric dipole at site of atom is called internal or local field. Thus, local or internal field at any point in the dielectric can be defined as the sum of the external field and all other effective fields produced by the induced electric dipoles.

To understand the local field, we consider that a dielectric material is placed between the parallel plate capacitor. Due to the presence of external electric field, atoms or molecules of dielectric materials get polarized. From Fig. 6, we can note that the field of consists of three components:

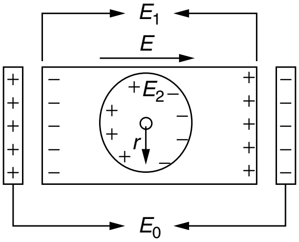


Figure 6 Uniform polarized dielectric between the plates of capacitor.

$$E_i = E_0 + E_1 + E_2 \tag{9.40}$$

E_0 is the external field due to the free charges and is given by

$$E_0 = \frac{\sigma_f}{\varepsilon_0} \tag{9.41}$$

E_1 is the field due to the bound charges on the outer surface of dielectric and is given by

$$E_1 = \frac{-\sigma_b}{\varepsilon_0} = \frac{P}{\varepsilon_0} \tag{9.42}$$

E_2 is the field at O due to bound charges on the surface of sphere and is given by

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3} \tag{9.43}$$

where q is the average bound charges and d is the vector from the negative center to the positive center. But we know that

$$qd = p = P \times V = P \times \frac{4}{3}\pi r^3 \tag{9.44}$$

So we have

$$E_2 = \frac{1}{4\pi\varepsilon_0}\frac{P(4/3)\pi r^3}{r^3} = \frac{P}{3\varepsilon_0} \tag{9.45}$$

Now substituting the values of E_0, E_1 and E_2 from Eqs. (9.41), (9.42) and (9.45), respectively, in Eq. (9.40) we get

$$E_i = \frac{D}{\varepsilon_0} - \frac{P}{\varepsilon_0} + \frac{P}{3\varepsilon_0} \tag{9.46}$$

$$\Rightarrow E_i = \frac{\varepsilon_0 E + P}{\varepsilon_0} - \frac{P}{\varepsilon_0} + \frac{P}{3\varepsilon_0} \qquad [D = \varepsilon_0 E + P] \tag{9.47}$$

$$\Rightarrow E_i = E + \frac{P}{3\varepsilon_0} \tag{9.48}$$

This is the required expression for internal field or local field. It is also known as *Lorentz field equation for local field.*

9.11 Clausius–Mossotti Equation

Clausius–Mossotti equation basically provides a relation between dielectric constant of a dielectric material with its atomic polarizibility. It has been observed that in the presence of local field E_i, each atom in polarized by an amount

$$p = \alpha E_i \tag{9.49}$$

where α is the polarizability of the atoms or molecules. According to the definition of polarization

$$P = Np = N\alpha E_i \tag{9.50}$$

From Eq. (9.48)

$$P = N\alpha\left[E + \frac{P}{3\varepsilon_0}\right] \tag{9.51}$$

Using relation

$$P = \varepsilon_0 E(\varepsilon_r - 1) \Rightarrow E = \frac{P}{\varepsilon_0(\varepsilon_r - 1)} \tag{9.52}$$

we get

$$P = N\alpha\left[\frac{P}{\varepsilon_0(\varepsilon_r - 1)} + \frac{P}{3\varepsilon_0}\right] \tag{9.53}$$

or

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{N\alpha}{3\varepsilon_0} \tag{9.54}$$

Equation (9.54) is known as Clausius–Mossotti equation.

In terms of Avogadro number

$$N = \frac{Na\rho}{M} \tag{9.55}$$

where ρ is the density and M is the molecular weight of the material. Then,

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{Na\rho}{M}\frac{\alpha}{3\varepsilon_0} \tag{9.56}$$

$$\Rightarrow \frac{M}{\rho}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = \frac{Na\alpha}{3\varepsilon_0} \tag{9.57}$$

Thus, Clausius–Mossotti equation relates microscopic polarization α to macroscopic dielectric constant (ε_r). It is applicable to gases and neutral liquids.

According to the electromagnetic theory, dielectric constant ε_r of the material is equal to the square root of the refractive index:

$$\varepsilon_r = \sqrt{n} \tag{9.58}$$

Thus, Clausius–Mossotti equation is also applicable to the optical frequency range and becomes

$$\frac{M}{\rho}\left(\frac{n^2-1}{n^2+2}\right) = \frac{N\alpha\alpha}{3\varepsilon_0} \tag{9.59}$$

9.12 Frequency Dependence of the Dielectric Constant

If the dielectric is subjected to an external alternating electric field, its polarization changes according to the field. The total polarization is the contribution of electronic, ionic and orientation polarizations which are individually frequency-dependent. Hence total polarization depends on how the atoms get oriented themselves with the applied field. The relative permittivity of dielectrics varies with the frequency of the field. When an electric field with certain frequency is applied to a dielectric, the electric dipoles are oriented themselves each time with the change of field. The average time taken by the dipoles to reorient themselves in the field direction in one complete cycle is known as **relaxation time** and reciprocal of the relaxation time is called **relaxation frequency**. The relaxation frequencies of different types of polarization mechanisms are different and any type of polarization mechanism will not continue if relaxation frequency of that is less than the frequency of electric field. The frequency-dependent dielectric constant curve is plotted in Fig. 7 which tells the existence of different types of polarization at different frequencies. Following are some points regarding the curve:

1. Below the frequency of 10^6 Hz, all types of polarization contribute in total polarization, thus polarization follows the field reversal as a result. The dipoles align themselves along the direction of external electric field.
2. In the frequency range $10^6 - 10^{11}$ Hz, dipoles fail to follow the field reversal, that is, orientational polarization ceases; thus the total polarization comes due to the ionic and electric polarization.

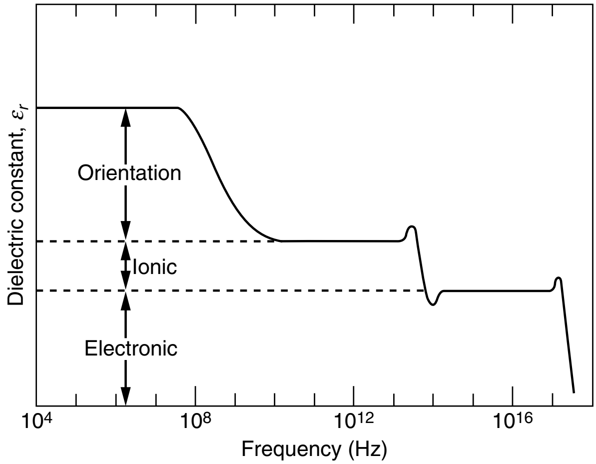


Figure 7 The variation of dielectric constant as a function of frequency.

3. In the frequency range $10^{11} - 10^{14}$ Hz, the contribution comes in the total polarization due to the electronic polarization.
4. For frequency > 10^{15} Hz, electric polarization fails to follow the field reversal; thus, the total polarization becomes zero and according to equation

$$\varepsilon_{\mathrm{r}} = 1 + \frac{P}{\varepsilon_0 E}$$

the dielectric constant becomes unity.

9.13 Dielectric Loss and Loss Tangent

To understand the phenomenon of dielectric loss, we consider the case of charging and discharging of a capacitor. When a capacitor is charged through potential V then the amount of energy received by the capacitor is (1/2) CV^2. This amount of energy is stored as electrostatic potential energy (or polarization energy) in the dielectric medium placed between the plates of the parallel plate capacitor. During the discharging of the capacitor, it is expected that the same amount of energy should be released. However, it has been observed that only a part of energy is released while rest is dissipated in the form of heat energy. *Thus, we can say that the amount of energy dissipated in the form of heat by a dielectric medium under the action of external electric field is known as dielectric loss.*

In order to determine dielectric losses we consider the permittivity as a complex number given by

$$\varepsilon^* = \varepsilon' - i\varepsilon'' \tag{9.60}$$

The real part ε' of Eq. (9.60) represents the ability of a material to be polarized by the action of external electric field. However, the imaginary part ε'' of the same equation is called loss factor.

We know that the applied field is

$$E = E_0 \exp(i\omega t) \tag{9.61}$$

where E_0 is the amplitude of the electric field, ω is the angular frequency and t is the time. Thus, polarization P of the dielectric under the action of external electric field is

$$P = \text{Real part of } (\varepsilon * E) \tag{9.62}$$

Now, substituting the values of ε^* and E from Eq. (9.60) and (9.61), respectively, in Eq. (9.62), we get

$$P = \text{Real part of } [(\varepsilon' - i\varepsilon'') E_0 \exp(i\omega t)] \tag{9.63}$$

$$\Rightarrow P = \text{Real part of } [(\varepsilon' - i\varepsilon'') E_0 (\cos\omega t + i \sin\omega t)]$$

$$\Rightarrow P = E_0 (\varepsilon' \cos\omega t + \varepsilon'' \sin\omega t) \tag{9.64}$$

Consider

$$P_0 \cos\delta = E_0\varepsilon' \tag{9.65}$$

and

$$P_0 \sin\delta = E_0\varepsilon'' \tag{9.66}$$

Substituting the values of $E_0\varepsilon'$ and $E_0\varepsilon''$ from Eqs. (9.65) and (9.66) in Eq. (9.64), we get

$$P = P_0 (\cos\omega t \cos\delta + \sin\omega t \sin\delta)$$

$$\Rightarrow P = P_0 \cos(\omega t - \delta) \tag{9.67}$$

Dividing Eq. (9.66) by Eq. (9.65) we get

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} \tag{9.68}$$

$\tan\delta$ is known as loss tangent. It represents the loss of electrical energy in terms of heat and, therefore, it is also called the dissipation factor. Now differentiating Eq. (9.67) with respect to time, then the current passing through the dielectric is given by

$$i = \frac{dP}{dt} = \frac{d}{dt}[P_0 \cos(\omega t - \delta)] \quad \text{or} \quad i = -\omega P_0 \sin(\omega t - \delta)$$

Suppose $i_0 = \omega\ P_0$. Then $i = -i_0 \sin(\omega t - \delta)$. Since power loss in dielectric is due to the imaginary part ε'' of the dielectric constant, therefore,

$$P = \varepsilon'' V_0 i_0 = \varepsilon'' \omega P_0 V_0 \Rightarrow P = \varepsilon'' \omega C V_0^2 \qquad [P_0 = CV_0]$$

$$\Rightarrow P = \varepsilon' \omega C V_0^2 \tan\delta \tag{9.69}$$

This is the required expression for dielectric loss.

9.14 Application of Dielectrics

Dielectrics are used in a number of electrical and electronics devices as per the property of the materials. Some examples where dielectrics are used are electrical bulbs, electrical insulators armature winding, iron electronics valves, switches and in capacitors.

Solved Examples

Example 1

Calculate the displacement vector D and electric permittivity in a dielectric material with dielectric constant 2.5 and polarization $P = 2.1 \times 10^{-8}$ C/m^2.

Solution: The polarization P of the material is related to its dielectric constant K or ε_r. Now

$$P = \varepsilon_0 E(\varepsilon_r - 1) \quad \text{and} \quad \varepsilon_0 E = \frac{D}{\varepsilon_r} \qquad \text{[using Eq. (9.30b)]}$$

So

$$D = \frac{P\varepsilon_r}{\varepsilon_r - 1}$$

Here ε_r or $K = 2.5$ and $P = 2.1 \times 10^{-8}$ C/m^2. So

$$D = \frac{2.1\times 10^{-8}\ \times 2.5}{2.5 - 1} = 3.5 \times 10^{-8}\ \text{C/m}^2$$

and

$$\varepsilon = \varepsilon_0 K = 2.5 \times 8.85 \times 10^{-12} = 22.13 \times 10^{-12}\ \text{C}^2\ \text{N}^{-1}\ \text{m}^{-2}$$

Example 2

The electrical susceptibility of material is 1.4×10^{-10} C^{2}N^{-1}m^{-2}. Find the dielectric constant and electric permittivity.

Solution: The dielectric constant of the material is given by

$$\varepsilon_r = 1+\frac{\chi_e}{\varepsilon_0} = 1+\frac{1.4\times10^{-10}}{8.85\times10^{-12}} = 15.82$$

$$\varepsilon = \varepsilon_0 K = 15.82\times8.85\times10^{-12} = 1.4\times10^{-10}\ \mathrm{C^2N^{-1}m^{-2}}$$

Example 3

At 0°C and 1 atmosphere pressure, the dielectric constant of helium is 1.000074. Determine the induced dipole moment on each helium atom and electric polarization when the gas is subjected to an electric field strength of 200 V/m.

Solution: Here, we have $K = 1.000074$, $N_a = 6.02\times10^{23}$, $V = 22.4$ liters $= 22.4\times10^{-3}\ \mathrm{m^3}$ and $E = 200$ V/m. Using Eq. (9.7) and (9.37), we get

$$P = \varepsilon_0 E(K-1) = Np$$

Therefore

$$p = \frac{\varepsilon_0 E(K-1)}{N} = \frac{\varepsilon_0 E(K-1)V}{N_a} \quad \left(\because N = \frac{Na}{V}\right)$$

Thus, induced dipole moment on each helium atom is

$$p = \frac{8.85\times10^{-12}\times200\times(1.000074-1)\times22.4\times10^{-3}}{6.02\times10^{23}} = 48.84\times10^{-40}\ \mathrm{Cm}$$

The electric polarization is

$$P = Np = 8.85\times10^{-12}\times200\times(1.000074-1) = 13.09\times10^{-14}\ \mathrm{Cm^{-2}}$$

Example 4

The dielectric constant of sulphur is 1.000025. Determine the induced dipole moment on each sulphur atom when the gas is subjected in electric field strength of 200 V/m. The atomic weight and density of sulphur are 32 and 2.08 g $\mathrm{cm^{-3}}$, respectively.

Solution: Here we have

$K = 1.000025$, $N_a = 6.02\times10^{23}$, $\rho = 2.08\ \mathrm{g\ cm^{-3}} = 2.08\times10^{3}\ \mathrm{kg\ m^{-3}}$, $M = 32$ and $E = 200$ V/m. Using Eqs. (9.7) and (9.37), we have

$$P = \varepsilon_0 E(K-1) = Np$$

Therefore

$$p = \frac{\varepsilon_0 E(K-1)}{N} = \frac{\varepsilon_0 E(K-1)M}{N_a\rho} \quad \left(\because N = \frac{N_a\rho}{M}\right)$$

Thus, induced dipole moment on each sulphur atom is

$$p = \frac{8.85\times10^{-12}\times200\times(1.000025-1)\times32}{6.02\times10^{23}\times2.08\times10^{3}} = 11.30\times10^{-40}\ \mathrm{Cm}$$

Example 5

A dielectric material having 0.6 cm^2 thickness and dielectric constant 7 is filled inside the plates separated by 1 cm and of area 100 cm^2. If developed potential difference between the plates is 200 V then find out the value of D, E and P.

Solution: Here we have $V_0 = 200$ V, $d = 1$ cm $= 10^{-2}$ m and $K = 7$. Electric field E_0 between the plates of the capacitor is

$$E_0 = \frac{V_0}{d} = \frac{200}{10^{-2}} = 2 \times 10^4 \text{ V/m}$$

Since $K = E_0/E$, therefore

$$E = \frac{2 \times 10^4}{7} = 2.85 \times 10^3 \text{ V/m}$$

Electric displacement is

$$D = K\varepsilon_0 E = 7 \times 8.85 \times 10^{-12} \times 2.85 \times 10^3 = 1.76 \times 10^{-7} \text{ Cm}^{-2}$$

and polarization is

$$P = (K - 1)\varepsilon_0 E = 6 \times 8.85 \times 10^{-12} \times 2.85 \times 10^3 = 1.51 \times 10^{-7} \text{ Cm}^{-2}$$

Example 6

Find the electrical susceptibility if dielectric constant of the insulator is 7 filled between the plates of the capacitor.

Solution: Here we have

$$\chi_e = \varepsilon_0(\varepsilon_r - 1) = 8.85 \times 10^{-12}\ (7 - 1) = 5.3 \times 10^{-11} \text{ C}^2\text{N}^{-1}\text{m}^{-2}$$

Example 7

Consider a point nucleus $+q$ surrounded by a uniformly charged spherical cloud $-q$ of radius a. Calculate the atomic polarizability of such an atom.

Solution: As we have discussed in the chapter, the nucleus will be shifted slightly to the right and electron cloud to the left in presence of an external electric field E. Now the field at a distance d from the center of a uniformly charged sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qd}{a^3} \qquad \text{or} \qquad p = qd = 4\pi\varepsilon_0 a^3 E$$

Since $\overline{p} = \alpha\overline{E}$, therefore

$$\alpha = 4\pi\varepsilon_0 a^3$$

Example 8

The number of atoms per unit volume of neon gas is 2.5×10^{26}. If the radius of the neon gas is 0.53 A determine the polarizability and relative permittivity.

Solution: We have

$$\alpha = 4\pi\varepsilon_0 a^3 = 4 \times 3.14 \times 8.85 \times 10^{-12} \times (0.53 \times 10^{-10})^3$$

$$= 1.6 \times 10^{-41} \text{ Fm}^2$$

The relative permittivity

$$\varepsilon_r = 1 + \frac{P}{\varepsilon_0 E} = 1 + \frac{N\alpha E}{\varepsilon_0 E}$$

Now substituting $\alpha = 4\pi\varepsilon_0 a^3$ and after solving we get

$$\varepsilon_\mathrm{r} = 1 + 4\pi N a_0^3 = 1 + 4 \times 3.14 \times 2.5 \times 10^{26} \times (0.53 \times 10^{-10})^3 = 1 + 0.0004 = 1.0004$$

Example 9

The atomic weight and density of sulphur are 32 and 2.08 g/(Cm)^3, respectively. Relative dielectric constant of the atom is 3.8. If sulphur solid has cubic symmetry, determine electronic polarazibility.

Solution: In terms of Avogadro's number, density and atomic weight, the Clausius–Mossotti equation becomes

$$\frac{M}{\rho}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right) = \frac{N_a \alpha}{3\varepsilon_0}$$

$$\Rightarrow \frac{32}{2.08 \times 10^3}\left(\frac{3.8 - 1}{3.8 + 2}\right) = \frac{6.02 \times 10^{26} \alpha}{3 \times 8.85 \times 10^{-12}}$$

After solving we get

$$\alpha = 3.28 \times 10^{-40} \text{ Fm}^2$$

Short Answers of Some Important Questions

1. What do you understand by dielectrics?

   Answer: Dielectrics are non-conducting materials like insulators.

2. What is the difference between dielectrics and insulators?

   Answer: There is a slight difference between dielectrics and insulators which is the function they perform. The main function of a dielectric material is to store electric energy in the form of potential energy while the function of an insulator is to obstruct of flow of current.

3. What is dielectric constant?

   Answer: Dielectric constant is the ratio of the capacitance of the capacitor filled with dielectric material to the capacitance of the capacitor with air medium for the same capacitor.

4. What do you understand by polarization?

   Answer: Induced dipole moment per unit volume is called polarization and the materials are said to be polarized.

5. What do you understand by polar and non-polar molecules?

   Answer: When centers of gravity of positive and negative charges coincide, the molecule is said to be **non-polar**. The non-polar molecules have zero permanent dipole moment due to symmetrical structure. Some example are H_2, N_2, Cl_2, O_2, CCl_4, BF_3, CO_2, etc. When the centers of gravity of positive and negative charges do not coincide, the molecule is said to be **polar**. The polar molecules have permanent dipole moment due to non-symmetrical structure. Some examples are H_2O, HCl, NH_3, CH_3Cl, CO, etc.

6. Define electric dipole and electric dipole moment?

   Answer: A pair of equal and opposite charges separated by a small distance is known as an electric dipole. However the product of magnitude of one of the charges and the distance between the two charges is called the dipole moment.

7. What is internal field in dielectric?

   Answer: Internal field or local field is the sum of external field and the field generated by dielectric molecules.

8. What is dielectric loss?

   Answer: The amount of energy dissipated in the form of heat by a dielectric medium under the action of external electric field is known as dielectric loss.

Important Points and Formulas

1. Dielectrics are non-conducting materials like insulators.
2. The main function of a dielectric material is to store electric energy in the form of potential energy while the function of an insulator is to obstruct of flow of current.
3. On the basis of the interaction of the field with the dielectric, the various types of dielectric are **linear**, **homogeneous** and **isotropic**.
4. Dielectric constant is the ratio of the capacitance of the capacitor filled with dielectric material to the capacitance of the capacitor with air medium for the same capacitor.
5. Induced dipole moment per unit volume is called polarization and the materials are said to be polarized:
   $$P = N \cdot p = N\alpha E$$
6. Electronic polarization occurs due to displacement of electron clouds from the nucleus.
7. Ionic polarization occurs due to the displacement of ions under the action of external electric field. For most of the materials the ionic polarizibility is less than the electronic polarizibility:
   $$\alpha_\mathrm{i} = \frac{1}{10}\alpha_\mathrm{e}$$
8. The orientational polarizability α_0 in terms of Boltzmann's constant k and absolute temperature T is
   $$\alpha_0 = \frac{p_0^2}{3kT}$$
9. Internal field $E_i = E + \dfrac{P}{3\varepsilon_0}$
10. Clausius–Mossotti equation $\dfrac{\varepsilon_\mathrm{r} - 1}{\varepsilon_\mathrm{r} + 2} = \dfrac{N\alpha}{3\varepsilon_0}$
11. The amount of energy dissipated in the form of heat by a dielectric medium under the action of external electric field is known as dielectric loss.

Multiple Choice Questions

1. Dielectric is
(a) conducting material
(b) non-conducting material
(c) semiconducting material
(d) None of these

2. The difference between dielectric and insulator is
(a) function
(b) electric field
(c) current
(d) None of the above

3. The relation between flux density and electric field is
(a) $D = \varepsilon_0 E$ **(b)** $D = \varepsilon/E$
(c) $D = \varepsilon E$ **(d)** $D = E/\varepsilon$

4. Which of the following is a dielectric medium?
(a) Aluminium **(b)** Copper
(c) HCl and Water **(d)** Germanium

5. HCl and water have permanent dipole moment even in absence of external electric field due to
(a) the difference in electronegativity
(b) internal field
(c) magnetic field
(d) None of these

6. In a dielectric, the polarization is
(a) exponential function of applied field
(b) logarithmic function of applied field
(c) linear function of applied field
(d) None of these

7. The unit of dipole moment per unit volume is
(a) C/m^2 **(b)** C/m^3
(c) C/m **(d)** None of these

8. The polarization that occurs in the frequency range 10^{12} Hz is
(a) electronic **(b)** ionic
(c) orientational **(d)** electronic and ionic

9. The relation between D, E and P is
(a) $D = \varepsilon_0 E + P$ **(b)** $D = \varepsilon E + P$
(c) $D = \varepsilon_0 E - P$ **(d)** None of these

10. Orientational polarization occurs in
(a) polar liquids
(b) non-polar liquids
(c) homogeneous solids
(d) heterogeneous solids

11. Electronic polarization occurs in
(a) ionic solids
(b) non-polar solids
(c) elemental solids
(d) All of these

12. The dielectric loss in a perfect dielectric is
(a) zero **(b)** infinite
(c) 100 **(d)** None of these

Short Answer Type Questions

1. What do you understand by dielectrics?

2. What are polar and non-polar molecules?

3. Define polarization in dielectric.

4. Name of different types of polarization.

5. What do you understand by internal fields?

6. What is displacement vector?

7. Write the relation between polarization and dielectric constant.

8. What do you understand by Clausius–Mossotti equation?

9. What is frequency dependence of dielectric constant?

10. What are dielectric loss and loss tangent?

Long Answer Type Questions

1. What do you understand by dielectrics? Differentiate between polar and non-polar dielectrics.
2. What happens when dielectric material is placed in external electric field? Explain it.
3. What do you mean by polarization of substance? Write different mechanisms of polarization in a dielectric.
4. What do you mean by polarization in dielectrics? Explain different types of polarization.
5. Discuss polar and non-polar molecules with suitable examples.
6. What are internal fields? Discuss Lorentz field equation for a non-polar isotropic dielectric.
7. Derive the Clausius–Mossotti relation. Discuss the physical significance of Clausius–Mossotti relation.
8. Discuss frequency dependence of dielectric constant and dielectric loss. Give some applications of dielectrics.

Numerical Problems

1. The dielectric constant of helium atom at 0°C and 1 atmosphere pressure is 1.0000686. Find the dipole moment induced in each helium atom when the gas is in an electric field of intensity 100 V/m.
2. The electrical susceptibility of a material is 27.4×10^{-12} $C^2N^{-1}m^{-2}$. What are the values of dielectric constant and the permittivity of the material? Also find the polarization if $D = 3 \times 10^{-8}$ C/m^2.
3. Determine the percentage of ionic polarizability in NaCl crystal which has the optical index of refraction and the static dielectric constant as 1.4 and 3.6, respectively.
4. Calculate the electronic polarizability of an Argon atom, given $K = 1.0023$ at NTP and $N = 2.7 \times 10^{25}$ atoms/m^3.

Answers

Multiple Choice Questions

1. (b)
2. (a)
3. (c)
4. (c)
5. (a)
6. (c)
7. (a)
8. (b)
9. (a)
10. (a)
11. (d)
12. (a)

Numerical Problems

1. 2.25×10^{-39} C-m
2. 4, 35.4×10^{-12} $C^2N^{-1}m^{-2}$ and 2.25×10^{-8} C/m^2
3. 48.31%
4. 7.54×10^{-40} Fm^2

10 Magnetic Properties of Materials

LEARNING OBJECTIVES

After reading this chapter you will be able to understand:

- Magnetization.
- Origin of magnetic moment.
- Diamagnetism, paramagnetism and ferromagnetism.
- Langevin's theory for diamagnetic material.
- Phenomena of hysteresis and its applications.

10.1 Introduction

Magnetism in a material arises due to the atomic orbital and spin motion of electrons. The electrons revolve in the orbit of atom which is equivalent to a tiny current loop and capable of producing magnetic field. Therefore, each atom of the material behaves as a small magnet with north and south poles and has a magnetic dipole moment. This magnetic dipole moment of the atom is not only because of the orbital motion but also due to the spin of electron. Normally in any material these atomic dipoles are randomly oriented with their magnetic dipole moments in different directions and hence there is no net magnetic moment in any direction which makes the material unmagnetized. However, in the presence of external magnetic field, the randomized nature of atomic dipoles is affected either by alignment or by the motion of electron. It all depends on the nature and property of the material.

On the basis of the magnetic behavior, materials are generally classified in the following three types:

1. Diamagnets.
2. Paramagnets.
3. Ferromagnets.

Since magnetization, magnetic dipole moment, magnetic susceptibility, permeability and magnetic intensity play an important role in studying the magnetic properties of materials, therefore first and foremost we have to discuss these before the classification of materials.

1. **Magnetic field (*B*):** The region around a magnet in which other magnets or an object made of magnetic material like iron, nickel or a moving charge particle experiences force is called a magnetic field. The unit of magnetic field is Weber/m^2.
2. **Magnetic intensity (*H*):** When a magnetic material is placed in a magnetic field, it becomes magnetized. The capability of the magnetic field to magnetize a material is said to be the magnetic intensity of the field. The unit of H is Ampere/meter.
3. **Magnetization (*I*):** When a ferromagnetic material is subjected to an external magnetic field, it experiences a magnetic dipole moment μ_m. This magnetic dipole moment per unit volume of the material is known as magnetization or intensity of magnetization. Mathematically,

$$I = \frac{\mu_m}{V} \tag{10.1}$$

Its unit is Ampere-turn/meter.

4. **Magnetic permeability:** It is the ratio of magnetic flux density B to the magnetic field intensity H and is denoted by μ. So

$$\mu = \frac{\vec{B}}{\vec{H}} \tag{10.2}$$

Its unit is Weber/ampere-meter or Henry/meter. For free space

$$\mu_0 = \frac{\vec{B_0}}{\vec{H}} \tag{10.3}$$

From Eqs. (10.2) and (10.3), we get

$$\frac{\mu}{\mu_0} = \frac{\vec{B}}{\vec{B_0}} = \mu_r \tag{10.4}$$

5. **Magnetic susceptibility:** When a magnetic material is subjected to an external magnetic field, the material gets magnetization which is proportional to the applied magnetizing field. Hence magnetic susceptibility is the ratio of intensity of magnetization I to the magnetizing field intensity H and is denoted by χ_m.

$$\chi_m = \frac{\vec{I}}{\vec{H}} \tag{10.5}$$

For vacuum $\chi_m = 0$.

10.2 Magnetic Dipole Moment due to an Electron: Bohr Magneton

We know that an electron not only spins, but also revolves in the orbits around the nucleus. This revolving electron can be treated equal to a tiny current loop. If an electron of charge $-e$ revolves in the orbit of radius r and completes a round in time T then current at any point in the loop can be expressed as

$$I = \frac{\text{Charge}}{\text{Time period}} = \frac{-e}{T} \tag{10.6}$$

$$\Rightarrow I = \frac{-e\omega}{2\pi} \quad \left(\because T = \frac{2\pi}{\omega}\right) \tag{10.7}$$

If A is the area of the current loop then orbital magnetic dipole moment is

$$p_m = IA = \frac{-e\omega}{2\pi} \times \pi r^2 = \frac{-e\omega r^2}{2} \tag{10.8}$$

If the electron with mass m is linearly moving with speed v, then orbital angular momentum acquired by the electron is

$$L = mvr = mr^2\omega \tag{10.9}$$

From Eq. (10.9) the orbital magnetic dipole moment is

$$p_{\mathrm{m}} = \frac{-eL}{2m} \tag{10.10}$$

Negative sign indicates induced magnetic dipole moment. According to Bohr's concept, angular momentum for revolving electrons should be equal to an integral multiple of $h/2\pi$:

$$p_{\mathrm{m}} = -\left(\frac{e}{2m}\right) n \frac{h}{2\pi} \Rightarrow p_{\mathrm{m}} = -n\left(\frac{eh}{4\pi m}\right) \tag{10.11}$$

The quantity $eh/4\pi m$ is known as a Bohr magneton (p_{B}):

$$p_{\mathrm{B}} = \left(\frac{eh}{4\pi m}\right) = 9.27 \times 10^{-24} \text{ Ampere-m}^2 \tag{10.12}$$

10.3 Classification of Materials

10.3.1 Diamagnetic Materials

There are certain magnetic materials whose atoms do not have permanent magnetic dipole moment. The electrons in the atomic orbit are generally in even numbers and moving in such a way that the magnetic moment of one electron is cancelled by the other. Hence net magnetic moment of atom and material is always zero. When the external magnetic field is applied to such materials, the speed of that electron which produces magnetic moment in opposite direction of applied field increases and the speed of other electron which produces magnetic moment in the direction of applied field decreases. Hence, the net magnetic moment is developed in the opposite direction of applied field and the material is weakly magnetized in the opposite direction of the applied magnetic field. Since there is no orientation of atomic dipole, therefore diamagnetism is independent of temperature. Further important points regarding diamagnetism are:

1. If these materials are placed in the presence of an external magnetic field, they are feebly magnetized in the direction opposite to that of the applied eternal magnetic field.
2. Since atomic number of these materials is generally even, therefore the net magnetic moment is zero.
3. Diamagnetism arises due to orbital motion of electrons.
4. Magnetic susceptibility of these materials is low and negative and also independent of temperature.
5. Relative permeability of these materials is less than 1.
6. Magnetic moment is low and in opposite direction to the applied magnetic field.
7. Ag, Cu, Au, Bi, etc. are examples of diamagnetic materials.

10.3.2 Paramagnetic Materials

Paramagnetic materials are those materials whose atoms have permanent magnetic dipole moment. In absence of any external magnetic field, the atomic dipoles are randomly oriented in different directions. As a result, net magnetic moment of these dipoles is zero. On the application of external magnetic field, these atomic dipoles are aligned to some extent in the direction of applied field and produce a weak net magnetic moment in the direction of applied magnetic field. Hence the material is weakly magnetized in the direction of applied magnetic field. Further important points regarding paramagnetism are:

1. If these materials are placed in the presence of an external magnetic field, they are feebly magnetized in the direction of the applied external magnetic field.

2. Since the atomic numbers of these materials is generally odd, therefore the net magnetic moment is not zero.
3. Paramagnetism arises due to spin motion of electrons.
4. Magnetic susceptibility of these materials is low but positive and inversely proportional to the absolute temperature (T):

$$\chi_{\mathrm{m}} = \frac{C}{T} \tag{10.13}$$

Here C is Curie constant.

5. Relative permeability of these materials is greater than 1.
6. Magnetic moment is low but along the direction of the applied magnetic field.
7. Mn, Na, Al, Pt, etc. are examples of paramagnetic materials.

10.3.3 Ferromagnetic Materials

The atoms of these materials like paramagnetic material have permanent magnetic dipole moment. The similar natures of dipoles are grouped in small region called domain. These domains have a net magnetic moment in a particular direction. In the material, there are large number of domains having magnetic moments in different directions which makes the net magnetic moment of entire material zero. When the external magnetic field is applied to such ferromagnetic materials, then either the domains are oriented in such a way so as to align with the direction of field or the size of favorable domain increases. Generally, in strong applied field the domains are aligned and in weak field the size of favorable domain increases. In both the cases, the material is strongly magnetized in the direction of applied external magnetic field. Further important points regarding ferromagnetism are:

1. If these materials are placed in the presence of an external magnetic field, they are strongly magnetized in the direction of the applied external magnetic field.
2. Due to unpaired electrons, the atoms of ferromagnetic materials have net magnetic dipole moment.
3. Ferromagnetism arises due to formation of domains.
4. Magnetic susceptibility of these materials is high and positive and also inversely proportional to the absolute temperature:

$$\chi_{\mathrm{m}} = \frac{C}{T - T_{\mathrm{c}}} \tag{10.14}$$

where T_{c} is Curie temperature.

5. Relative permeability of these materials is much greater than 1.
6. Magnetic moment is high but along the direction to the applied magnetic field.
7. Fe, Ni, Co, etc. are examples of paramagnetic materials.

10.4 Langevin's Theory of Diamagnetism

The theory of diamagnetism was given by Langevin in 1905, on the basis of electron theory using the fact that diamagnetism arises due to the orbital motion of electrons. We have already discussed in Section 9.5.1 that the electron revolves around the nucleus and revolving of electron is equivalent to the current loop. So the current at any point in the loop is expressed as

$$I = \frac{\text{Charge}}{\text{Time period}} = -\frac{e}{T} \tag{10.15}$$

But $T = 2\pi/\omega$. Hence,

$$I = -\frac{e\omega}{2\pi} \tag{10.16}$$

If A is the area of the current loop then orbital magnetic moment is given as

$$p_{\mathrm{m}} = IA$$

Now

$$I = -\frac{e\omega}{2\pi} \times \pi r^2$$

where r is the radius of current loop
So

$$p_{\mathrm{m}} = -\frac{e\omega\, r^2}{2} \tag{10.17}$$

Since electron revolves in a circular orbit, therefore centripetal force acts along the radial inward direction

$$F = \frac{mv^2}{r} = mr\omega^2 \quad [v = r\omega] \tag{10.18}$$

Under the action of an external magnetic field, an additional force (i.e. Lorentz force) acts on the electron in the radial direction (see Fig. 1); this force is denoted by F_{m} and is mathematically expressed as

$$F_{\mathrm{m}} = evB = er\omega B \tag{10.19}$$

where e is the charge on electron, r is the radius of current loop, ω is angular frequency and B is the magnetic flux density.

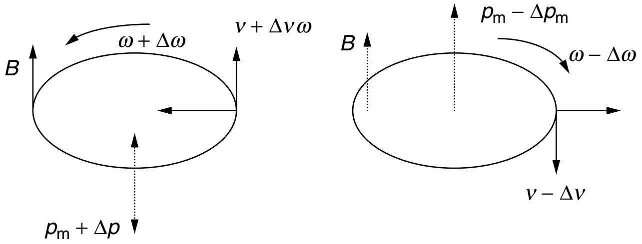


Figure 1 Lorentz force.

Thus, the resultant force acting on the revolving electron is

$$F' = F \pm F_{\mathrm{m}} \tag{10.20}$$

where '+' ve sign stands for radially inward and '–' ve for radially outwards directions. If $\Delta\omega$ is a charge in angular frequency under the action of external magnetic field, then

$$F' = mr(\omega + \Delta\omega)^2 \tag{10.21}$$

Using Eqs. (10.19), (10.20) and (10.21), we get

$$mr(\omega + \Delta\omega)^2 = mr\omega^2 + er\omega B$$

$$\Rightarrow mr\omega^2 + mr\Delta\omega^2 + 2mr\omega\ \Delta\omega = mr\omega^2 + er\omega B$$

Since, $\omega >> \Delta\omega$, $\Delta\omega^2$ can be neglected. So

$$mr\omega^2 + 2mr\omega\ \Delta\omega = mr\omega^2 + er\omega B$$

$$\Rightarrow \Delta\omega = \frac{eB}{2m} \tag{10.22}$$

Now, change in magnetic moment in presence of an external magnetic field is

$$\Delta p_{\mathrm{m}} = -\frac{1}{2} e\Delta\omega r^2$$

$$\Rightarrow \Delta p_{\mathrm{m}} = -\frac{1}{4}\frac{e^2 r^2}{m} B$$

Thus, the total magnetic moment in the presence of external magnetic field is

$$p_{\mathrm{m}} = \sum \Delta p_{\mathrm{m}} \quad \text{or} \quad p_{\mathrm{m}} = \sum\left(-\frac{1}{4}\frac{e^2 r^2}{m} B\right) \tag{10.23}$$

If n is the total number of atoms/unit volume, then magnetization of unit volume is

$$I = np_{\mathrm{m}} = -\frac{ne^2 B}{4m}\sum r^2 \tag{10.24}$$

But we know that $B = \mu_0 H$ and hence

$$I = -\frac{ne^2 \mu_0 H}{4m}\sum r^2 \tag{10.25}$$

$$\Rightarrow \frac{I}{H} = -\frac{ne^2 \mu_0}{4m}\sum r^2$$

According to the definition of susceptibility

$$\chi_{\mathrm{m}} = \frac{I}{H} = -\frac{ne^2 \mu_0}{4m}\sum r^2 \tag{10.26}$$

Equation (10.26) is the required expression for the susceptibility of diamagnetism. From the above equation we can say that the susceptibility of diamagnetic material is negative and temperature independent.

10.5 Hysteresis

When a ferromagnetic material is subjected to an external gradually increasing magnetic field, the magnetization of the material increases in the direction of the field and reaches a saturation point. It decreases with the decrease of external field but retains some magnetization even if the field is completely removed. Now when the direction of external field is reversed and it increases gradually, the material magnetizes in the reverse direction. After reaching saturation point of magnetization, if the external field is further reduced to bring it at zero value, the material still retains some magnetization in the direction opposite to the previous one. So it can be said that *the magnetization is always lagging behind the field. This lagging of magnetization behind magnetic field is called hysteresis and a curve plotted between the magnetization I and magnetizing field H is known as hysteresis curve* (see Fig. 2). In complete round of the magnetization of the material, some energy is left in the material which is later lost as heat. This loss of energy is known as hysteresis loss which is equal to the area enclosed by the closed hysteresis curve or loop.

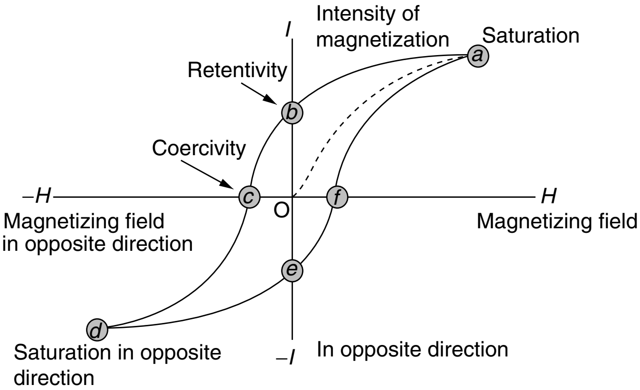


Figure 2 Hysteresis curve.

In this context, a ferromagnetic material is placed in a magnetizing field H. If we increase H, then magnetization I increases but not uniformly and goes to maximum value of I at a point a as shown in Fig. 2. At point a, intensity of magnetization does not increase on further increasing H. This is the condition of magnetic saturation. If H now decreases, I also decreases but it follows a new path *ab* instead of original path *oa*. When H reduces to zero, I still has some value which is equal to *ob* and is known as retentivity (the magnetism that remains in the material after removing the magnetizing field is called retentivity or rememance). If the direction of the magnetizing field is increased in negative direction of the field then magnetization traces the path *bc* and becomes zero at point *c*; the value *oc* is called coercivity or coercive force of the material. Further, if we increase H, saturation point is also observed. Finally, if we reverse the direction of the magnetizing field, the specimen traces the path along *defa*. Thus, the closed loop *abcdefa* is obtained which is called the hysteresis curve (see Fig. 2).

10.6 Hysteresis Loss

In ferromagnetic materials the spontaneously magnetized domains play an important role for the magnetization of the specimen. In presence of an external magnetic field, the domains of these materials align themselves along the direction of the applied magnetic field. In this alignment process, the specimen absorbs some energy; one can expect that the same amount of energy should be released when we remove the magnetizing field. *However, the supplied energy to the specimen is not fully recovered on removing of field and remains in the specimen. This unrecovered energy which may be later lost as heat is called hysteresis loss and equal to the area of hysteresis loop.*

10.7 Hysteresis Loss in *B*–*H* Curve

Let us consider a ring of ferromagnetic material having N turns, cross-sectional area A, permeability μ and circumferential length l. If the coil produces magnetic induction B in presence of the current I then the total flux associated with the coil is

$$\phi_m = NBA \tag{10.27}$$

The magnetic field due to the current is

$$H = \frac{Ni}{l} \quad \Rightarrow i = \frac{Hl}{N} \tag{10.28}$$

We know that induced e.m.f. is given by

$$e = -\frac{d\phi_m}{dt} = -NA\frac{dB}{dt} \tag{10.29}$$

The work done $d\omega$ in presence of the current is

$$d\omega = eidt \tag{10.30}$$

Now putting the values of e and i from Eqs. (10.29) and (10.28) in Eq. (10.30), we get

$$d\omega = AlHdB$$

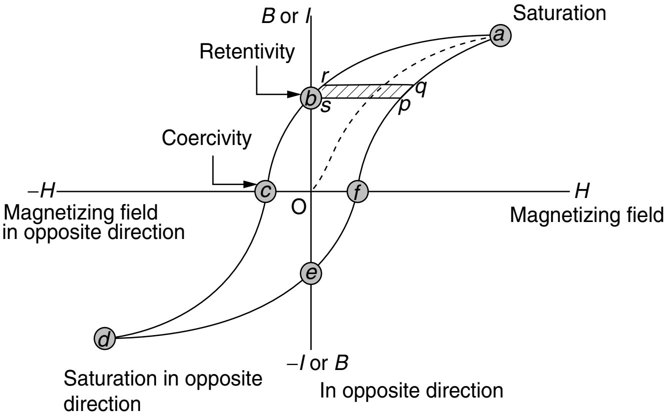


Figure 3 Hysteresis loss in B–H (or I–H) curve.

Since, Al is the volume, hence the work done per unit volume

$$dw = \frac{AlHdB}{Al} = HdB \tag{10.31}$$

From Fig. 3 area of the shaded strip $pqrs$ is HdB. Thus, the work done per unit volume for a complete cycle (hysteresis loss) is given by integrating Eq. (10.31) from B_1 to B_2:

$$w = \int_{B_1}^{B_2} dw$$

$$\text{Hysteresis loss} = HdB = \text{Area of the } B\text{–}H \text{ curve} \tag{10.32}$$

10.8 Hysteresis Loss in *I*–*H* Curve

We know that

$$B = \mu_0(H + I) \tag{10.33}$$

Differentiating the above equation, we get

$$dB = \mu_0(dH + dI)$$

Multiplying by H we get

$$HdB = \mu_0 HdH + \mu_0 HdI$$

If we plot a graph between H and dH, straight line can be obtained and the area covered by the straight line is zero. Hence,

$$HdB = \mu_0 HdI$$

Thus, hysteresis loss in I–H curve is

$$\text{Hysteresis loss} = HdI = \mu_0 \times \text{ Area of the } I - H \text{ curve} \tag{10.34}$$

10.9 Comparison between Soft Iron and Steel

We have compared the magnetic properties of steel and soft iron through Fig. 4 on the basis of their hysteresis curve.

1. The area of the hysteresis in steel is more, therefore hysteresis loss is high; in soft iron the area of the hysteresis is less, therefore hysteresis loss is low.
2. The value of retentivity in steel is low as compared to soft iron; however the value of coercivity in steel is higher than observed in soft iron.
3. The value of permeability is low in steel but high in soft iron.
4. The value of susceptibility is also low in steel but high in soft iron.

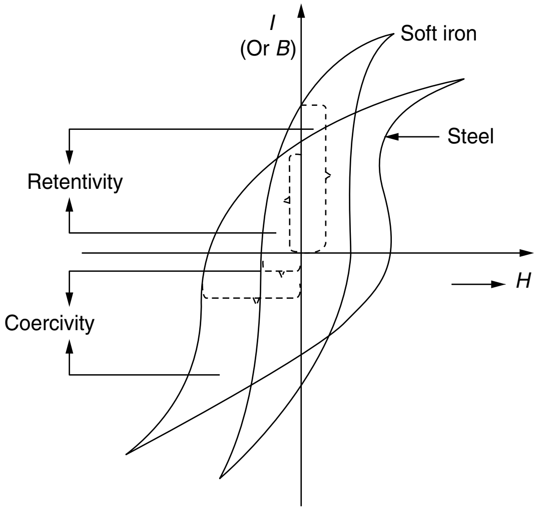


Figure 4 Comparison between steel and soft iron.

10.10 Use of Hysteresis Curve

Hysteresis curve is useful in choosing the material for making permanent magnets, electromagnets and transformer cores.

10.10.1 *Permanent Magnets*

The material used for permanent magnet should have high saturation magnetization and large value of coercivity so that these are not easily demagnetized. Steel is used for making permanent magnets due to the low value of retentivity, permeability, susceptibility and high value of coercivity.

10.10.2 Electromagnets

Soft iron is used for making electromagnets due to the high value of retentivity, permeability, susceptibility and low value of coercivity.

10.10.3 Transformer Cores

Soft iron is also used for making transformer cores due to the high value of retentivity, permeability, susceptibility and low value of coercivity.

Solved Examples

Example 1

A bar magnet has a coercivity of 7×10^3 A/m. It is desired to demagnetize it by inserting it inside a solenoid 20 cm long and having 50 turns. What current should be sent through the solenoid?

Solution: We have $H = 7 \times 10^3$ A/m, $l = 20$ cm = 0.2 m and $N = 50$ turns. Now we have

$$H = Ni/l$$

or

$$I = Hl/N = 7 \times 10^3 \times 0.2/50 = 28\ A$$

Example 2

A magnetic material has a magnetization of 3000 A/m and flux density 0.005 Weber/m^2. Determine the magnetizing force and the relative permeability of material.

Solution: We have $I = 3000$ A/m, $\mu_0 = 4\pi \times 10^{-7}$ Henry/m and $B = 0.005$ Weber/m^2. Since $B = \mu_0 (I + H)$, we have

$$H = \frac{B}{\mu_0} - I = \frac{0.005}{4\pi \times 10^{-7}} - 3000 = 980 \text{ A/m}$$

We know that relative permeability μ_r is given by

$$\mu_r = \frac{I}{H} + 1 = \frac{3000}{980} + 1 = 4.06$$

Short Answers of Some Important Questions

1. What is diamagnetic substancs?

   Answer: The substances which when placed in a magnetic field, acquire feeble magnetized in the direction opposite to that of the external magnetic field are known as diamagnetic substances.

2. What do you understand by magnetic susceptibility?

   Answer: When a magnetic material is subjected to an external magnetic field, the material gets magnetization which is proportional to the applied magnetizing field. Hence magnetic susceptibility is the ratio of intensity of magnetization I to the magnetizing field H and is denoted by χ_m.

$$\chi_m = \frac{\vec{I}}{\vec{H}}$$

3. What is intensity of magnetization?

Answer: When a ferromagnetic material is subjected to an external magnetic field, it experiences a magnetic dipole moment μ_m. This magnetic dipole moment per unit volume of the material is known as magnetization or intensity of magnetization. Mathematically,

$$I = \frac{\mu_m}{V}$$

Its unit is Ampere-turn/meter.

4. What is hysteresis loss?

Answer: In complete round of the magnetization of the material, some energy is left in the material which is later lost as heat. This loss of energy is known as hysteresis loss which is equal to the area enclosed by the closed hysteresis curve or loop.

5. What do you understand by hysteresis and hysteresis curve?

Answer: The magnetization is always lagging behind the field. This lagging of magnetization behind magnetic field is called hysteresis and a curve plotted between the magnetization I and magnetizing field H is known as hysteresis curve.

6. What is retentivity?

Answer: The magnetism that remains in the material after removing the magnetizing field is called retentivity or rememance.

Important Points and Formulas

1. Magnetism in a material arises due to the atomic orbital and spin motion of electrons.
2. The quantity $eh/4\pi m$ is known as a Bohr magneton.
3. If diamagnetic materials are placed in the presence of an external magnetic field, they are feebly magnetized in the direction opposite to that of the applied eternal magnetic field.
4. If paramagnetic materials are placed in the presence of an external magnetic field, they are feebly magnetized in the direction of the applied eternal magnetic field.
5. If paramagnetic materials are placed in the presence of an external magnetic field, they are strongly magnetized in the direction of the applied eternal magnetic field.
6. The required expression for the susceptibility of diamagnetism.

$$\chi_m = \frac{I}{H} = -\frac{ne^2\mu_0}{4m}\sum r^2$$

7. The susceptibility of diamagnetic material is negative and temperature independent.
8. In complete round of the magnetization of the material, some energy is left in the material which is later lost as heat. This loss of energy is known as hysteresis loss which is equal to the area enclosed by the closed hysteresis curve or loop.
9. Hysteresis loss = HdB = Area of the B–H curve.

Multiple Choice Questions

1. The relation between B, M and H is

(a) $B = \mu_0(H + M)$ **(b)** $H = \mu_0(B + M)$

(c) $B = \mu_0(H - M)$ **(d)** $B = \mu H$

2. The susceptibility of diamagnetic materials is

(a) positive **(b)** negative

(c) more positive **(d)** None of these

3. The susceptibility of paramagnetic materials is

(a) positive **(b)** negative

(c) large **(d)** small

4. The susceptibility of ferromagnetic materials is

(a) small **(b)** large

(c) very large **(d)** negative

5. Spontaneous polarization occurs in
   (a) paramagnetic materials
   (b) dimagnetic materials
   (c) ferromagnetic materials
   (d) antiferromagnetic materials

6. Hysteresis loss occurs in
   (a) AC voltage
   (b) DC voltage
   (c) Both (a) and (b)
   (d) None of these

7. The unit of magnetic flux is
   (a) Wb/m
   (b) Wb/m^2
   (c) Wb
   (d) Ampere

Short Answer Type Questions

1. What is magnetism?

2. Define Curie temperature for ferromagnetism.

3. Mention two properties of diamagnetic materials.

4. What is the advantage of hysteresis?

5. Distinguish between diamagnetic and paramagnetic substances.

6. Why transformer cores are made of soft iron?

Long Answer Type Questions

1. What are diamagnetic, paramagnetic and ferromagnetic substances? Establish the relation $B = \mu_0 (I + H)$.

2. Differentiate between diamagnetic, paramagnetic and ferromagnetic substances.

3. Discuss Langevin's theory of diamagnetism.

4. Show that susceptibility of diamagnetic material is negative and independent of temperature.

5. What is meant by hysteresis? Explain hysteresis loss. How would you use the hysteresis curves to select the material for the construction of permanent magnet?

6. What is hysteresis? Discuss the importance of this curve.

7. Differentiate between steel and soft iron.

Numerical Problems

1. A magnetic material has a magnetization of 3000 A/m and flux density 0.005 Weber/m^2. Determine the magnetizing force and the permeability of material.

2. An iron rod 0.2 m long, 10 mm in diameter and of relative permeability 100 is placed inside a long solenoid wound with 300 turns/meter. If a current of 0.5 A is passed through the solenoid, find the magnetic moment of the rod.

3. A bar magnet has a coercivity of 5×10^3 A/m. It is desired to demagnetize it by inserting it inside a solenoid 10 cm long and having 50 turns. What current should be sent through the solenoid?

Answers

Multiple Choice Questions

1. (a) **2.** (b) **3.** (a) **4.** (c) **5.** (c) **6.** (a) **7.** (b)

Numerical Problems

1. 980 A/m and 4.06 **2.** 0.233 A-m^2 **3.** 10 A

11 Electromagnetics

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Displacement current.
- Equation of continuity.
- Maxwells equations (integral and differential forms).
- Poynting vector and Poynting theorem.
- Electromagnetic wave equation and its propagation characteristics in free space.
- Non-conducting and in-conducting media.
- Skin depth.

11.1 Introduction

You have already studied static (i.e. time independent) electric and magnetic fields in electrostatics and magnetostatics, respectively in previous classes. These fields are produced by the charges at rest and steady currents respectively, and can be analyzed independently. But if these fields vary with time, one cannot analyze them independently. Now the question arises: Why? The answer is: Faraday's law of electromagnetic induction shows that a time-varying magnetic field produces an electric field while Ampere's law shows that a time-varying electric field produces a magnetic field. Thus, changing of electric and magnetic field with time, a field of other kind is induced in the adjacent space which produces electromagnetic waves consisting electric and magnetic fields.

11.1.1 Laws of Electromagnetics Before Maxwell

There are four basic laws of electricity and magnetism before Maxwell which are as follows:

1. Gauss' law of electrostatics

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \quad \text{or} \quad \oint \vec{E}\cdot d\vec{S}=\frac{q}{\varepsilon_0}$$

Here q is charge and ρ is the volume charge density.

2. Gauss' law of magnetostatics

$$\vec{\nabla}\cdot\vec{B}=0 \quad \text{or} \quad \oint \vec{B}\cdot d\vec{S}=0$$

3. Faraday's law of induction

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \text{or} \quad \oint \vec{E}\cdot d\vec{l}=-\frac{\partial\phi_B}{\partial t}$$

where ϕ_B is the magnetic flux.

4. Ampere's law

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \quad \text{or} \quad \oint \vec{B}\cdot d\vec{l}=\mu_0 I=\mu_0\int_S \vec{J}\cdot d\vec{S}$$

Here I is current and J is current density.

These equations are the relation between the field and their source and are used to solve the problems of electromagnetic theory since long, even before the Maxwell started his work. Among the above four equations, the Ampere's law in the present form is true only for steady case. Maxwell noticed this inconsistency in equation during his study while applying Ampere's law to a capacitor. Thus, Maxwell formulated the concept of displacement current to remove this inconsistency and modified the Ampere's law which will be discussed in following sections.

11.2 Displacement Current

The concept of displacement current was first conceived by Maxwell to explain the production of magnetic field in empty space. According to him, it is not only the current in a conductor that produces a magnetic field, but a changing electric field in a vacuum or in a dielectric also produces a magnetic field. This means that a changing electric field is equivalent to a current and gives same effect to magnetic field as the conduction current. This equivalent current is known as *displacement current* which exists in the space as long as the electric field is changing and is expressed as

$$\varepsilon_o \frac{d\phi_E}{dt}$$

In order to explain the displacement current mathematically, we consider the case of parallel-plate capacitor. Let at any particular instant, q be the charge on capacitor plate. According to the definition, conduction current at any instant is

$$i_c = \frac{dq}{dt} \tag{11.1}$$

We have already discussed about electrical displacement ($D = \varepsilon_o E$) in dielectrics (Chapter 9). Therefore,

$$D = \sigma = \frac{q}{A} \tag{11.2}$$

where σ is the surface charge density and A is the area of the parallel-plate capacitor. From Eq. (11.2) we have

$$q = DA \tag{11.3}$$

Now substituting the value of q from Eq. (11.3) in Eq. (11.1), we get

$$i_c = \frac{d}{dt}(DA) = A\frac{dD}{dt} \tag{11.4}$$

Maxwell suggested that the term $i_d = \varepsilon_o \frac{d\phi_E}{dt}$ should be considered as the current inside the dielectric. This current is called as displacement current and is denoted by i_d. Hence,

$$i_d = \varepsilon_o \frac{d\phi_E}{dt} \quad \Rightarrow \quad i_d = A\varepsilon_o \frac{dE}{dt} \quad \Rightarrow \quad i_d = A\frac{dD}{dt} \quad \Rightarrow \quad i_d = EA \tag{11.5}$$

We know that $J_d = i_d / A$ is current density. Therefore Eq. (11.5) may be written in terms of current density J_d as

$$\vec{J}_d = \frac{d\vec{D}}{dt} \tag{11.6}$$

or

$$\vec{J}_d = \varepsilon_0 \frac{dE}{dt} \qquad [\because \vec{D} = \varepsilon_0 E] \tag{11.7}$$

Thus, the current arising due to time-varying electric field between the plates of a capacitor is called the displacement current.

11.2.1 Characteristics of Displacement Current

1. Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current because it is not related to the motion of charges.
2. Inside the dielectric there will be a displacement current which is equal to conduction current.
3. Displacement current is only an apparent current representing the rate at which flow of charge takes place from one plate to another plate.
4. Displacement current in good conductors is almost nil as compared to conduction current below the frequency 10^{15} Hz.

11.3 Equation of Continuity

Continuity equation is the consequence of conservation of charge. Law of conservation of charges states that electric charges can neither be created nor destroyed. Therefore, the total current flowing out of the system of some volume must be equal to the rate of decrease of charge within the volume. Therefore, when the current flows at any region of volume V, bounded by a closed surface S then

$$i = -\frac{dq}{dt} = \oint_S \vec{J} \cdot d\vec{S} \tag{11.8}$$

But we know that total charge is enclosed by the close surface in terms of volume charge density ρ with in volume V, that is,

$$q = \int_V \rho \, dV \tag{11.9}$$

Therefore

$$i = -\frac{dq}{dt} = \oint_S \vec{J} \cdot d\vec{S} = -\int_V \frac{\partial \rho}{\partial t} dV$$

or

$$\oint_S \vec{J} \cdot d\vec{S} + \int_V \frac{\partial \rho}{\partial t} dV = 0 \tag{11.10}$$

From the fundamental theorem of divergence, which is a relation between surface integral to volume integral, we have

$$\oint_S \vec{J} \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{J} \, dV$$

$$\Rightarrow \int_V \left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right) dV = 0$$

which is true for any arbitrary volume, therefore,

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \tag{11.11}$$

This equation is the continuity equation and is based on the conservation of charge. When we use time derivative term

$$\frac{\partial\rho}{\partial t}=0$$

the above continuity equation is reduced to

$$\vec{\nabla}\cdot\vec{J}=0 \tag{11.12}$$

That is, the net flux of current through any closed surface is zero which is the case of steady state.

11.4 Modification of Ampere's Law

1. **Integral form of Ampere's law:** Maxwell modified the Ampere's law by introducing the term of displacement current from the study of charging and discharging of a capacitor. If we look at the simple circuit with a capacitor C in Fig. 1, the current flows in the circuit after proper connection, the charges start accumulating on the capacitor plates and the magnetic field between the plates as well as outside plate (around wire) is observed. As there is no actual flow of charges between plates, there is no conduction current as well, but the electric field in space due to charges on plates continuously changes with the time as long as the charges on plates change. This changing electric field cause the generation of magnetic field between the plates.

   Now in Fig. 1, we consider a small loop around the wire just to analyze the magnetic field due to conduction current i in wire, then according to present Ampere's law "The line integral of magnetic induction B around a closed path is equal to μ_0 times the current enclosed by the path." Mathematically

$$\oint\vec{B}\cdot d\vec{l}=\mu_o i \tag{11.13}$$

   If the loop encloses a surface area S_1 then according to Stokes' theorem

$$\oint\vec{B}\cdot d\vec{l}=\int_{S_1}(\vec{\nabla}\times\vec{B})\cdot d\vec{S}=\mu_o i \tag{11.14}$$

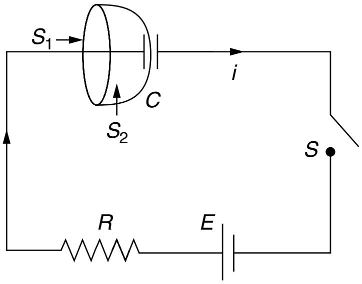


Figure 1 Modification in Ampere's law.

But if the loop encloses a surface area S_2 (according to fundamental theorem of curl, i.e. Stokes' theorem, no matter what surface you consider, if it is bounded with the same loop), no conduction current passes through this surface. Then,

$$\oint \vec{B}\cdot d\vec{l} = \int_{S_2} (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = 0 \tag{11.15}$$

The above two equations for the same loop with different surfaces are not same and the right-side values of equations contradict, therefore, both cannot be true. Hence, the present form of Ampere's law is inconsistent or not true for all cases.

Now from the definition of displacement current which is

$$i_d = \varepsilon_o \frac{d\phi_E}{dt}$$

which is developed in the space between capacitor plate at surface S_2 and equal to the conduction current in magnitude. Hence, either the conduction or the displacement current is present at any surface under consideration, therefore both currents are to be considered in the Ampere's law and equation is modified in following form:

$$\oint \vec{B}\cdot d\vec{l} = \mu_o\left(i + \varepsilon_o \frac{d\phi_E}{dt}\right) = i + i_d \tag{11.16}$$

Now after modification of equation as in above case, when S_1 surface is considered, i_d is absent and if S_2 surface is considered, the only i_d is present and anomaly or inconsistency in equation is removed.

2. To look at the differential form of Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I = \mu_0 \int_S \vec{J}\cdot d\vec{S} \tag{11.17}$$

where $\vec{J}$ is the current density in the conductor having cross-sectional area $\vec{S}$. Using Stokes' law which is a relation between line integral and surface integral, we have

$$\oint \vec{B}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0 \int_S \vec{J}\cdot d\vec{S} \tag{11.18}$$

Since surface is arbitrary, so we have

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} \qquad \text{or} \qquad \vec{\nabla}\times\vec{H} = \vec{J} \tag{11.19}$$

Taking divergence on both sides of Eq. (11.19), we have

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J}$$

Since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0$, then also

$$\vec{\nabla}\cdot\vec{J} = 0 \tag{11.20}$$

Equation (11.20) is valid only for steady current. For other non-steady cases $\vec{\nabla}\cdot\vec{J} \neq 0$. In other words, J is not always a solenoidal vector, hence Eq. (11.19) is inconsistent.

Also from the equation of continuity

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \qquad \text{or} \qquad \frac{\partial\rho}{\partial t}=0$$

Here ρ is constant that shows charge density is not changing with the time. As a result, Ampere's law should be modified for time-varying field using a quantity $\vec{J}_D$ which is to be added to the right-hand side of Eq. (11.19), so that J together with $\vec{J}_D$ becomes the solenoidal vector whose divergence is always zero. Therefore, the following equation after introducing $\vec{J}_D$ is true for all cases.

$$\vec{\nabla}\times\vec{H}=\vec{J}+\vec{J}_D \tag{11.21}$$

It can be explained in the following way: The equation of continuity

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$$

from differential form of Gauss law (first Maxwell equation)

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0} \qquad \text{or} \qquad \rho=\vec{\nabla}\cdot\vec{D}$$

Then

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\vec{\rho}}{\partial\vec{t}}=\vec{\nabla}\cdot\vec{J}+\vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial\vec{t}}=\vec{\nabla}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial\vec{t}}\right)=0 \tag{11.22}$$

Here $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ is the solenoidal vector whose divergence is always zero. To remove the inconsistency in Ampere's law, Maxwell suggested that the current density $\vec{J}$ should be replaced by $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ in Eq. (11.19). Hence, by introducing the term $\vec{J}+(\partial\vec{D}/\partial\vec{t})$ in Eq. (11.19), the following is the correct modified differential form of Ampere's law which is true for time varying as well as for steady currents.

$$\vec{\nabla}\times\vec{H}=\vec{J}+\left(\frac{\partial\vec{D}}{\partial t}\right)$$

or

$$\vec{\nabla}\times\vec{H}=\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{11.23}$$

11.5 Maxwell's Equations

Maxwell, in 1864, theoretically derived the connection between the charges at rest (electrostatics), charges in motion (current electricity), electric and magnetic field (electromagnetic) and summarized in terms of four equations: Gauss' law in electrostatic, Gauss' law in magnetostatics, Ampere's law and Faraday's laws. These equations are called Maxwell's equations. Table 1 gives the four Maxwell's equations in differential and integral forms.

Table 1 Maxwell's equations in differential and integral form

S. No.	*Differential Form*	*Integral Form*
1.	$\operatorname{div}\vec{E}=\dfrac{\rho}{\varepsilon_0}$ or $\vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0}$	$\oint\vec{E}\cdot d\vec{S}=\dfrac{q}{\varepsilon_0}$ or $\oint\vec{E}\cdot d\vec{S}=\dfrac{1}{\varepsilon_0}\int\rho\,dV$
2.	$\operatorname{div}\vec{B}=0$ or $\vec{\nabla}\cdot\vec{B}=0$	$\oint\vec{B}\cdot d\vec{S}=0$
3.	$\operatorname{curl}\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$ or $\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$	$\oint\vec{E}\cdot d\vec{l}=-\dfrac{\partial\phi_B}{\partial t}$
4.	$\operatorname{curl}\overrightarrow{H}=\vec{J}+\left(\dfrac{\partial\vec{D}}{\partial t}.\right)$ or $\vec{\nabla}\times\overrightarrow{H}=\vec{J}+\left(\dfrac{\partial\vec{D}}{\partial t}.\right)$	$\oint\overrightarrow{H}\cdot d\vec{l}=\int_S\left(\vec{J}+\dfrac{\partial\overrightarrow{D}}{\partial t}\right)\cdot d\vec{S}$
	where ρ is the charge density. $\vec{D}=\varepsilon_0\vec{E}$, electric displacement vector, ε_0 is the permittivity of the free space and $\vec{E}$ is the electric field strength. $\vec{B}=\mu_0\vec{H}$, where μ_0 is the magnetic permeability of free space and $\vec{H}$ is the magnetic field intensity.	

11.5.1 Derivation of Maxwell's First Equation

According to Gauss' law in electrostatics 'The net flux passing through a closed surface is equal to $1/\varepsilon_0$ times the total charge q contained in the volume enclosed by surface.' Mathematically,

$$\phi_E=\oint\vec{E}\cdot d\vec{S}=\frac{q}{\varepsilon_0}\tag{11.24}$$

where $\oint\vec{E}\cdot d\vec{S}$ represents the total flux passing through closed surface S. But we know that total charge enclosed in the surface in terms of volume charge density ρ with in volume V is

$$q=\int_V\rho\,dV\tag{11.25}$$

From Eqs. (11.24) and (11.25), we get that

$$\oint\vec{E}\cdot d\vec{S}=\frac{1}{\varepsilon_0}\int\rho\,dV$$

or

$$\int_S\overrightarrow{D}\cdot d\vec{S}=\int_V\rho\,dV\qquad(\because\overrightarrow{D}=\varepsilon_0\overrightarrow{E})$$

By Gauss' divergence theorem

$$\int_S\overrightarrow{D}\cdot d\vec{S}=\int_V(\operatorname{div}\overrightarrow{D})\cdot d\overrightarrow{V}=\int_V\rho\,dV$$

$$\Rightarrow\int_V(\operatorname{div}\overrightarrow{D}-\rho)\,dV=0$$

Since volume is arbitrary, hence

$$\operatorname{div}\vec{D} - \rho = 0 \qquad \text{or} \qquad \operatorname{div}\vec{D} = \rho \tag{11.26}$$

In free space, volume charge density ρ is zero. Therefore, Maxwell's first equation in free space is

$$\operatorname{div}\vec{D} = \vec{\nabla}\cdot\vec{D} = 0 \tag{11.27}$$

11.5.2 Maxwell's Second Equation

We know that magnetic monopole does not exist in the nature. Since magnetic lines of force entering or leaving a closed surface are equal, therefore, the net magnetic flux passing through the area $d\vec{S}$ of a closed surface S is zero:

$$\int_S \vec{B}\cdot d\vec{S} = 0 \tag{11.28}$$

Using Gauss' divergence theorem which is a relation between surface integral to volume integral as given below

$$\oint_S \vec{B}\cdot d\vec{S} = \int_V (\vec{\nabla}\cdot\vec{B})\, dV$$

$$\int_V (\vec{\nabla}\cdot\vec{B})\, dV = 0$$

Since the volume is arbitrary

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{11.29}$$

This is the requirement of Maxwell's second equation and it is true for free as well as material medium.

11.5.3 Maxwell's Third Equation

According to Faraday's law of electromagnetic induction the induced electromagnetic force around a closed circuit is equal to the negative time rate of charge of magnetic flux linked with the circuit. Thus,

$$e = -\frac{d\phi}{dt} \tag{11.30}$$

But we know that

$$e = \oint_c \vec{E}\cdot d\vec{l} = -\frac{d\phi}{dt}$$

$$\Rightarrow \oint_c \vec{E}\cdot d\vec{l} = -\frac{d\int_S \vec{B}\cdot d\vec{S}}{dt} = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} \tag{11.31}$$

From Stokes' fundamental theorem

$$\oint_c \vec{E}\cdot\vec{dl} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S}$$

$$\Rightarrow \int (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

Since surface S is arbitrary, hence

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{11.32}$$

This is Maxwell's third equation for free as well as for material medium.

11.5.4 Maxwell's Fourth Equation

The integral form of Maxwell fourth equation is

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \left(i + \varepsilon_o \frac{d\phi}{dt} \right) \tag{11.33}$$

The current i in term of J and electric flux ϕ in terms of E for any surface can be expressed as

$$i = \int \vec{J} \cdot d\vec{S}$$

$$\phi = \int \vec{E} \cdot d\vec{S}$$

The right side of equation can be expressed as

$$i + \varepsilon_o \frac{d\phi}{dt} = \int \vec{J} \cdot d\vec{S} + \varepsilon_o \int \frac{d\vec{E}}{dt} \cdot d\vec{S} \tag{11.34}$$

From Stokes' theorem

$$\oint \vec{B} \cdot d\vec{l} = \int \left(\vec{\nabla} \times \vec{B} \right) \cdot d\vec{S} \tag{11.35}$$

Therefore

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \left[\int \vec{J} \cdot d\vec{S} + \varepsilon_o \int \frac{d\vec{E}}{dt} \cdot d\vec{S} \right]$$

$$= \mu_0 \left[\int \left(\vec{J} + \varepsilon_o \frac{d\vec{E}}{dt} \right) \cdot d\vec{S} \right] \tag{11.36}$$

Since surface S is arbitrary, hence

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_o \frac{d\vec{E}}{dt} \right)$$

$$\vec{\nabla} \times \vec{H} = \left(\vec{J} + \frac{d\vec{D}}{dt} \right) \qquad [\because (\vec{D} = \varepsilon_o \vec{E})] \tag{11.37}$$

This is Maxwell's fourth equation in differential form.

11.6 Maxwell's Equation in Integral Form

1. Maxwell's first equation in differential form is

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{11.38}$$

Integrating it with respect to volume V, we get

$$\int_V (\vec{\nabla} \cdot \vec{D})\, dV = \int_V \rho\, dV$$

The volume integral can be changed into surface integral with the help of Gauss divergence theorem as

$$\int_V \left(\vec{\nabla} \cdot \vec{D} \right) dV = \oint \vec{D} \cdot d\vec{S} \tag{11.39}$$

$$\oint \vec{D}\cdot d\vec{S} = \int_V \rho\, dV$$

or
$$\oint \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0} \tag{11.40}$$

This is the integral form of Maxwell's first equation.

2. Maxwell's second equation in differential form is

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{11.41}$$

Integrating second equation with respect to the volume V, we get

$$\int_V (\vec{\nabla}\cdot\vec{B})\, dV = 0$$

By Gauss' divergence theorem

$$\int_V (\vec{\nabla}\cdot\vec{B})\, dV = \oint \vec{B}\cdot d\vec{S}$$

$$\oint \vec{B}\cdot d\vec{S} = 0 \tag{11.42}$$

where S is the surface enclosing volume V. This is the integral form of Gauss' divergence theorem in magnetostatics.

3. Maxwell's third equation in differential form is

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{11.43}$$

Integrating the above equation over an open surface S, we get

$$\int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{S} = -\int_S \left(\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S}$$

The surface integral can be converted into line integral through Stokes' theorem as

$$\int_S \left(\vec{\nabla}\times\vec{E}\right)\cdot d\vec{S} = \oint \vec{E}\cdot d\vec{l} = -\int_S \left(\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S}$$

Therefore
$$\oint \vec{E}\cdot d\vec{l} = -\int_S \left(\frac{\partial\vec{B}}{\partial t}\right)\cdot d\vec{S}$$

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial\phi_B}{\partial t} \qquad \left(\because \phi_B = \int_S \vec{B}\cdot d\vec{S}\right) \tag{11.44}$$

This is the integral form of Faraday's law of electromagnetic induction. Equation (11.44) is the integral form of Maxwell's third equation.

4. Maxwell's fourth equation in differential form is

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \tag{11.45}$$

Integrating above equation with respect to S, we get

$$\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{S} = \int_S \left(\vec{J}+\frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{S}$$

From Stokes' theorem,

$$\int_S (\overrightarrow{\nabla}\times\overrightarrow{H})\cdot d\vec{S} = \oint \overrightarrow{H}\cdot d\vec{l} = \int_S \left(\overrightarrow{J}+\frac{\partial \overrightarrow{D}}{\partial t}\right)\cdot d\vec{S}$$

$$\oint \overrightarrow{H}\cdot \overrightarrow{dl} = \int_S \vec{J}\cdot d\vec{S} + \int_S \frac{\partial \overrightarrow{D}}{\partial t}\cdot d\vec{S}$$

$$\oint \overrightarrow{H}\cdot \overrightarrow{dl} = \left(i+\varepsilon_o \frac{d\phi_E}{dt}\right) \quad \left(\because \phi_E = \int_S \overrightarrow{E}\cdot d\vec{S}\right) \tag{11.46}$$

Equation (11.46) is the integral form of Maxwell's fourth equation.

11.7 Physical Significance of Maxwell's Equations

11.7.1 Maxwell's First Electromagnetic Equation

Because of time independence, *Maxwell's first electromagnetic equation* is a steady-state equation. It represents the Gauss' law in electrostatics which states that the electric flux through any closed hypothetical surface is equal to $1/\varepsilon_0$ times the total charge enclosed by the surface.

11.7.2 Maxwell's Second Electromagnetic Equation

Maxwell's second electromagnetic equation represents Gauss' law in magnetostatics. It states that the net magnetic flux through any closed surface is zero (i.e., the number of magnetic lines of flux entering any region is equal to the lines of flux leaving it). It also explains that no isolated magnetic pole exists.

11.7.3 Maxwell's Third Electromagnetic Equation

Maxwell's third electromagnetic equation represents Faraday's law in electromagnetic induction. It states that an electric field is induced in the form of close lines when magnetic flux (or lines of magnetic force) changes through an open surface. The line integral of induced electric field around a close path is equal to the negative rate of change of magnetic flux.

11.7.4 Maxwell's Fourth Electromagnetic Equation

Maxwell's fourth electromagnetic equation represents the modified form of Ampere's circuital law which states that a changing electric field produces a magnetic field and an electric field can also be produced by changing magnetic field. Therefore, *Maxwell's fourth electromagnetic equation* gives the new concept of generation of magnetic field by displacement current.

11.8 Poynting Vector and Poynting Theorem

The moving oscillating coupled electric and magnetic fields behave as electromagnetic waves. These waves are transverse in nature where electric and magnetic vectors oscillate perpendicular to the direction of motion. During propagation, these waves also transport energy and momentum. The waves, when strike any surface, exert a pressure on the surface.

Poynting Vector is defined as the energy transported by wave per unit area per unit time. It is denoted by a vector P and can be expressed by the cross product of electric and magnetic field in the following way

$$\vec{P} = \vec{E} \times \vec{H} \qquad \text{or} \qquad \vec{P} = \frac{\vec{E} \times \vec{B}}{\mu_o} \tag{11.47}$$

The direction of the flow of this power through unit area is in the direction of propagation of wave. Its SI unit is Watt/m^2.

Poynting theorem is a work–energy theorem of electromagnetics and expressed as *work done on the charges by the electromagnetic forces is equal to the decrease in energy stored in the fields, and less than the energy that flows out through the surface*. To derive and explain the Poynting theorem, let us take third and fourth Maxwell equations as follows:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.48}$$

$$\vec{\nabla} \times \overrightarrow{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{11.49}$$

Taking the dot product of $\vec{H}$ with Eq. (11.48) and that of $\vec{E}$ with Eq. (11.49), we have

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{11.50}$$

$$\vec{E} \cdot (\vec{\nabla} \times \overrightarrow{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \tag{11.51}$$

Subtracting Eq. (11.50) from Eq. (11.51), we get

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \overrightarrow{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \vec{J} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right] - \vec{E} \cdot \vec{J}$$

From vectors product

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \overrightarrow{H}) = \vec{\nabla} \cdot (\vec{E} \times \overrightarrow{H}) \tag{11.52}$$

Therefore,

$$\vec{\nabla} \cdot (\vec{E} \times \overrightarrow{H}) = -\left[\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right] - \vec{E} \cdot \vec{J} \tag{11.53}$$

But $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$. Therefore

$$\nabla \cdot (\vec{E} \times \overrightarrow{H}) = -\left[\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\right] - \vec{E} \cdot \vec{J}$$

$$= -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] - \vec{E} \cdot \vec{J} \quad \left(\because \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) \text{and } \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right)$$

$$\vec{E} \cdot \vec{J} = -\left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] - \nabla \cdot (\vec{E} \times \overrightarrow{H})$$

Taking the volume integral over a volume V enclosed by surface S, we get

$$\int_V (\vec{E}\cdot\vec{J})\,dV = -\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)\right] dV - \int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV \tag{11.54}$$

Using Gauss divergence theorem

$$\int_V \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dV = \oint(\vec{E}\times\vec{H})\cdot dS$$

Hence,

$$\int_V \left(\vec{E}\cdot\vec{J}\right) dV = -\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)\right] dV - \oint(\vec{E}\times\vec{H})\cdot dS \tag{11.55}$$

Equation (11.55) represents the work energy theorem of electromagnetic and is called Poynting theorem for the flow of energy in an electromagnetic field.

1. The term $\int_V (\vec{E}\cdot\vec{J})\,dV$ represents the work done per unit time on the charges by electromagnetic fields.
2. The term $-\int_V \left[\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right)\right] dV$ represents the rate of decrease of stored energy in electric and magnetic fields in volume V.
3. $-\oint(\vec{E}\times\vec{H})\cdot dS$ represents the rate of flow of energy through surface area S enclosing volume V.

Here $\vec{P} = \vec{E}\times\vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector.

11.9 Plane Electromagnetic Waves in Free Space

We describe one of the important applications of Maxwell's equations to derive electromagnetic wave equations for field vectors $\vec{E}$ and $\vec{B}$. In free space, where there is no charge or current (i.e. $\rho = 0$, $\vec{J} = 0, \varepsilon = \varepsilon_0$, $\mu = \mu_0$, $B = \mu_0 H$ and $D = \varepsilon_0 E$), Maxwell's equations are as follows:

$$\vec{\nabla}\cdot\vec{E} = 0 \tag{11.56}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{11.57}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{11.58}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{11.59}$$

Taking the curl on both sides of Eq. (11.58) we get

$$\vec{\nabla}\times(\vec{\nabla}\times E) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$

Using Eq. $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ and $\vec{\nabla}\cdot\vec{E} = 0$

$$\nabla^2 E = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} \tag{11.60}$$

Similarly taking the curl of fourth equation (11.59) we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2 B = -\mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

Using $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ and $\vec{\nabla}\cdot\vec{B} = 0$ we have

$$\nabla^2 B = \mu_0\varepsilon_0\frac{\partial^2 B}{\partial t^2} \tag{11.61}$$

In vector form

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}, \qquad \nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2} \tag{11.62}$$

The general wave equation for any function like u moving with speed v is

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{11.63}$$

Therefore, from the above equation, Eq. (11.62) represents wave equations for ***E*** and ***B*** in free space. Each Cartesian component of ***E*** and ***B*** satisfies the three-dimensional wave equation.

So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed

$$\mu_0\varepsilon_0 = \frac{1}{v^2} \Rightarrow v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{(4\pi\times10^{-7}\ \text{Weber/A-m})(8.85\times10^{-12}\ \text{C}^2\text{N-m}^2)}} = 2.99\times10^{8}\ \text{m/s}$$

Hence, *electromagnetic waves propagate in free space with the speed of light*:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{11.64}$$

In some other medium, velocity is given as

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{(\mu_0\mu_r)(\varepsilon_0\varepsilon_r)}} \tag{11.65}$$

where μ_r and ε_r are relative permeability and relative permittivity, respectively. Using Eq. (11.64), Eq. (11.65) can be written as

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

or

$$v = \frac{c}{\sqrt{\varepsilon_r}} \qquad \text{[For non-magnetic material } \mu_r = 1\text{]}$$

As we know that the refractive index n of the medium is

$$n = \frac{c}{v}$$

Therefore,

$$n = \sqrt{\varepsilon_r} \tag{11.66}$$

The speed of light in a material is always less than in vacuum because ε_r has a value greater than one.

11.10 Transverse Nature of Electromagnetic Waves

The electromagnetic waves are transverse in nature where E and B vector oscillate perpendicular to the propagation direction. To explain the transverse nature, let us have the solution of wave equations which are mathematically second order differential equations. The equations are

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \quad \text{and} \quad \nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0 \tag{11.67}$$

The general solution of these equation are respectively

$$\vec{E}(r,t) = \vec{E}_o e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

and

$$\vec{B}(r,t) = \vec{B}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{11.68}$$

where $\vec{E}_o$ and $\vec{B}_0$ are the complex amplitudes for electric and magnetic fields, respectively, whose real part represent the physical value. $\vec{k}$ is the wave vector and $\vec{r}$ is position vector which are expressed as

$$\vec{k} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi}{c/\upsilon}\hat{n} = \frac{2\pi\upsilon}{c}\hat{n} = \frac{\omega}{c}\hat{n}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\hat{n}$ is unit vector represents the wave propagation direction. Then

$$\vec{k}\cdot\vec{r} = (k_x\hat{i} + k_y\hat{j} + k_z\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = (k_x x + k_y y + k_z z)$$

Now considering solution $\vec{E}$, we find the divergence of Eq. (11.56), that is, $\bar{\nabla} \cdot \bar{E} = 0$.

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{k}E_{0z} \right) e^{i(k_x x + k_y y + k_z z - \omega t)} \\
&= i(E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)e^{i(k_x x + k_y y + k_z z - \omega t)} \\
&= i(\hat{i}k_x + \hat{j}k_y + \hat{h}k_z) \cdot (\hat{i}E_{0x} + \hat{j}E_{0y} + \hat{h}E_{0z})e^{i(k_x x + k_y y + k_z z - \omega t)} \\
&= i[\vec{k} \cdot E_0 e^{i(k_x x + k_y y + k_z z - \omega t)}] \\
&= i\vec{k} \cdot \vec{E}
\end{aligned}
$$

Since $\vec{\nabla} \cdot \vec{E} = 0$. So

$$\vec{k} \cdot \vec{E} = 0 \tag{11.69}$$

or $\vec{E}$ is perpendicular to $\vec{k}$. Now $\vec{k}$ has direction of wave propagation, so $\vec{E}$ is perpendicular to the direction of propagation. Similarly, consider second equation (11.57), $\vec{\nabla} \cdot \vec{B} = 0$. We get

$$\vec{\nabla} \cdot \vec{B} = i(\vec{k} \cdot \vec{B}) \Rightarrow \vec{k} \cdot \vec{B} = 0. \tag{11.70}$$

So $\vec{B}$ is perpendicular to the direction of wave propagation. Therefore EM wave is transverse in nature.

11.11 Characteristic Impedance

Consider Maxwell's third equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$$

We solve it by considering $\vec{E}$ and $\vec{B}$ as given by Eq. (11.68):

$$
\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_{0x}e^{i(\vec{k}\cdot\vec{r} - \omega t)} & E_{0y}e^{i(\vec{k}\cdot\vec{r} - \omega t)} & E_{0z}e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{vmatrix}
$$

$$
\begin{aligned}
&= \hat{i}\left[\frac{\partial}{\partial y}\left(E_{0z}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) - \frac{\partial}{\partial z}\left(E_{oy}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) \right] \\
&\quad + \hat{j}\left[\frac{\partial}{\partial z}\left(E_{0x}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) - \frac{\partial}{\partial x}\left(E_{oz}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) \right] \\
&\quad + \hat{k}\left[\frac{\partial}{\partial x}\left(E_{0y}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) - \frac{\partial}{\partial y}\left(E_{ox}e^{i(k_x x + k_y y + k_z z - \omega t)} \right) \right] \\
&= i\left[\hat{i}(k_y E_{0z} - k_z E_{0y}) + \hat{j}(k_z E_{0x} - k_x E_{0z}) + \hat{k}(k_x E_{0y} - k_y E_{0x}) \right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}
\end{aligned}
$$

$$= i[\vec{k} \times \vec{E}_0]e^{i(\vec{k}\cdot\vec{r}-\omega t)} = i[\vec{k} \times \vec{E}]$$

and
$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(B_0 e^{i(k_x x+k_y y+k_z z-\omega t)} = -i\omega B_0 e^{i(k_x x+k_y y+k_z z-\omega t)} = -i\omega\vec{B}$$

Hence from $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, we have

$$i(\vec{k}\times\vec{E}) = i\omega\vec{B} \quad\Rightarrow\quad \vec{k}\times\vec{E} = \omega\vec{B} \tag{11.71}$$

Similarly from Maxwell's fourth equation $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$, we have

$$\vec{k}\times\vec{B} = -\omega\mu_0\varepsilon_0\vec{E} \tag{11.72}$$

From Eqs. (11.71) and (11.72) it can be concluded that electric and magnetic vectors $\vec{E}$ and $\vec{B}$ are mutually perpendicular to each other and perpendicular to the direction of propagation vector $\vec{k}$ (see Fig. 2). Further from Eq. (11.71) we have

$$\vec{k}\times\vec{E} = \omega\vec{B}$$

$$k(\hat{n}\times\vec{E}) = \omega\vec{B}$$

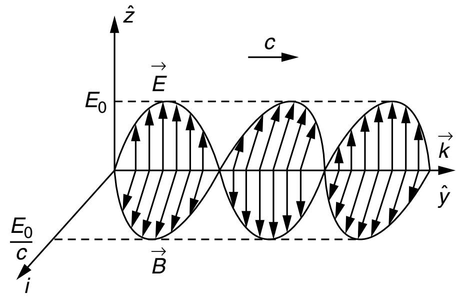


Figure 2

The mod of the above equation is

$$k\left|\hat{n}\times\vec{E}\right| = \omega\left|\vec{B}\right|$$

$$\frac{\vec{E}}{\vec{B}} = \frac{\omega}{k} = c \qquad \left(k = \frac{\omega}{z}\right)$$

or
$$\frac{\vec{E}}{\vec{H}} = \mu_0 c = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad [\vec{B} = \mu_0 H]$$

E/H is the characteristic impedance or intrinsic impedance of free space denoted by Z_0 and has the unit electrical resistance. Its value is

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{45\times10^{-7}}{8.86\times10^{-12}}} = 376.7\,\Omega \tag{11.73}$$

This implies that electric vector $\vec{E}$ and magnetic field vector $\vec{B}$ are in the same phase.

11.12 Electromagnetic Waves in Dielectric Medium

Since we are familiar that there is no free charge in dielectric medium therefore, $\rho = 0$, $\sigma = 0$, and hence $\vec{J} = \sigma E = 0$. However, μ and ε have finite values. So Maxwell's equations are as follows:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{11.74}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11.75}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.76}$$

$$\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \tag{11.77}$$

Taking curl on both sides of Eq. (11.76) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Using Eq. (11.77) we have

$$\nabla^2 E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad [\because \vec{\nabla} \cdot \vec{E} = 0]$$

Similarly taking curl of Eq. (11.77) we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu\varepsilon \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 B = -\mu\varepsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

Using Eq. (11.76) we have

$$\nabla^2 B = \mu\varepsilon \frac{\partial^2 B}{\partial t^2} \qquad [\because \vec{\nabla} \cdot \vec{B} = 0]$$

In vector form

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} \tag{11.78}$$

Equation (11.78) represents wave equations for ***E*** and ***B*** in dielectric medium.

11.13 Electromagnetic Waves in Conducting Medium

In conducting medium, the charge given to material is always lie at the surface and no charge stay inside the conducting material, hence charge density $\rho = 0$. So, for a conducting medium Maxwell's equations are as follows:

$$\vec{\nabla} \cdot \vec{D} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In conducting medium $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{J} = \sigma \vec{E}$, where σ represents the conductivity of the isotropic and homogeneous medium. Thus, Maxwell's equations reduced to

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{11.79}$$

$$\vec{\nabla} \cdot B = 0 \tag{11.80}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{11.81}$$

$$\vec{\nabla} \times \vec{B} = \mu\left(\sigma\ \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right) \tag{11.82}$$

To derive wave equation in conducting medium take the curl on both sides of Eq. (11.81), we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

Substituting the value of $\vec{\nabla} \times \vec{B}$ from Eq. (11.82) in the above equation, we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 E = -\mu \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right)$$

Since $\vec{\nabla} \cdot \vec{B} = 0$ is from Maxwell's first equation, we have

$$-\nabla^2 E = -\mu\sigma \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

or

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{11.83}$$

Similarly, we can obtain wave equation for ***B*** by taking curl of Eq. (11.82) and using Eq. (11.80) as

$$\nabla^2 B - \mu\sigma \frac{\partial B}{\partial t} - \mu\varepsilon \frac{\partial^2 B}{\partial t^2} = 0 \tag{11.84}$$

The above equations are wave equations in conducting medium. If we take $\sigma = 0$ and permeability and permittivity for free space, the above equations will be for free space. In conducting medium, the wave vector k is a complex and the real part of it determines the physical values of wave such as wavelength and speed of wave. The imaginary part of wave vector results in an attenuation of wave (decreasing amplitude of E and B with depth of penetration in medium). Here unlike in free space, the electric and magnetic field vectors are no longer in phase, rather magnetic field lags behind the electric field (Fig. 3).

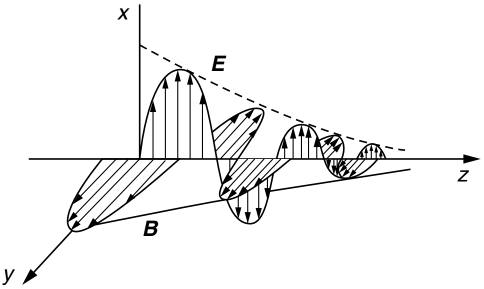


Figure 3 Phase-diagram of electric and magnetic field vectors.

11.14 Skin Depth

Skin depth is an essential parameter for the wave when the electromagnetic waves penetrate in conducting medium. *It is the depth in conducting medium in which the strength of electric field is reduced* $1/e$ *times of its original values.*

The skin depth is frequency dependent for good conductor and frequency independent for poor conductor. Consider the solution of wave equation (11.83) as

$$E(r,t) = E_o e^{j(k.r-\omega t)}$$

where k is complex and can be expressed with real and imaginary term α and β respectively as $k = \alpha + j\beta$. Now if wave is moving along z direction with E vector parallel to x, then E will be

$$E_x(z,t) = E_{ox} e^{j((\alpha+j\beta)z-\omega t)}$$

or

$$E_x(z,t) = E_{ox} e^{-\beta z} e^{j(\alpha z-\omega t)}$$

the attenuation factor is $e^{-\beta z}$. E_x should be $(1/e)$ times its original value if $\beta z = 1$. In this case z, the depth in the medium becomes skin depth and is denoted by δ as shown in Fig. 4. Hence,

$$z = \delta = \frac{1}{\beta} \tag{11.85}$$

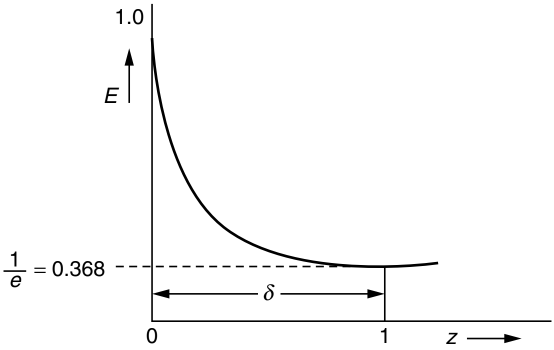


Figure 4 Skin depth.

The value of α and β can be obtained with wave equation

$$\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

and its solution

$$E_x(z,t) = E_{ox} e^{j[(\alpha + j\beta)z - \omega t]}$$

which will be

$$\alpha = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2} \right)^{1/2} + 1 \right]}; \quad \beta = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2} \right)^{1/2} - 1 \right]} \tag{11.86}$$

For good conductor $(\sigma \gg \varepsilon\omega)$, Hence

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \left(\frac{\sigma}{\omega\varepsilon} \right)} \quad \text{or} \quad \beta = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Thus, skin depth

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{11.87}$$

In terms of frequency (f) skin depth will be

$$\delta = \sqrt{\frac{2}{2\pi f \mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}} \tag{11.88}$$

From Eq. (11.88), we can conclude that *skin depth or penetration depth is inversely proportional to the root of frequency of wave.*

For poor conductor $(\sigma \ll \varepsilon\omega)$

$$\beta = \omega \sqrt{\sqrt{\frac{\mu\varepsilon}{2}} \left[\left(1 + \frac{1}{2} \frac{\sigma^2}{\omega^2\varepsilon^2} \right) - 1 \right]}$$

or

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2} \times \frac{1}{2} \frac{\sigma^2}{\omega^2\varepsilon^2}}$$

or

$$\beta = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}$$

Thus, skin depth

$$\delta = \frac{1}{\beta} = \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu}} \tag{11.89}$$

From the above equation, we can conclude that *skin depth or penetration depth is independent of frequency of wave.*

Solved Examples

Example 1

Prove that electromagnetic waves propagate with speed of light.

Solution: The wave equations for $\vec{E}$ and $\vec{B}$ in free space are as follows:

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

In vacuum, then, each Cartesian component of $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfies the three dimensional wave equation. Hence

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, travelling at a speed

$$\mu_0 \varepsilon_0 = \frac{1}{v^2}$$

$$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{(4\pi \times 10^{-7} \text{ Weber/A-m})(8.85 \times 10^{-12} \text{ C}^2\text{N-m}^2)}} = 2.99 \times 10^8 \text{ m/s}$$

Hence, *electromagnetic waves propagate with the speed of light*:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

From this result we can conclude that light is an electromagnetic wave.

Example 2

Prove that the speed of light in a material is always less than that in vacuum.

Solution: We know that in vacuum material travels with velocity of light. In some other medium, velocity is given as

$$v = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{(\mu_0 \mu_r)(\varepsilon_0 \varepsilon_r)}}$$

where μ_r and ε_r are relative permeability and relative permittivity, respectively. Since

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

therefore

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

or $$v = \frac{c}{\sqrt{\varepsilon_r}} \qquad \text{[For non-magnetic material } \mu_r = 1\text{]}$$

The speed of light in a material is always less than that in vacuum because ε_r has a value greater than one.

Example 3

Determine refractive index and velocity of light if the relative permittivity of distilled water is 64.

Solution: The velocity of distilled water is given by

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

where $\mu_r = 1$, $c = 3 \times 10^8$ m/sec and $\varepsilon_r = 64$. Therefore

$$v = \frac{3 \times 10^8}{\sqrt{64}} = 3.75 \times 10^7 \text{ m/s}$$

As we know, the refractive index n of the medium is $n = c/v$. Therefore,

$$n = \sqrt{\varepsilon_r} = \sqrt{64} = 8$$

Example 4

A uniform plane wave having electric field intensity in air as 7×10^3 V/m in the y-direction is propagating in the x-direction at a frequency of 2×10^8 rad/sec. Determine the frequency, wavelength, time-period and amplitude of ***H***.

Solution: We have

$$E_y = 7 \times 10^3 \cos(2 \times 10^8 t - px)$$

Here $\omega = 2 \times 10^8$ rad/sec, $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\varepsilon_0 = 8.85 \times 10^{-12}$ C^{2}N-m^2. Now frequency is given by

$$\upsilon = \frac{w}{2\pi} = \frac{2 \times 10^8}{2 \times 3.14} = 318.5 \times 10^5 \text{ Hz} = 3.18 \times 10^7 \text{ Hz}$$

Wavelength is given by

$$\lambda = \frac{v}{\upsilon} = \frac{3 \times 10^8}{3.18 \times 10^7} = 9.43 \text{ m}$$

Time period is given by

$$\lambda = \frac{1}{\upsilon} = \frac{1}{3.18\times 10^{7}} = 3.14\times 10^{-8} \text{ sec}$$

Amplitude of ***H*** is

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377\ \Omega$$

$$\Rightarrow H = \frac{7\times 10^{3}}{377} = 18.56 \text{ A/m}$$

Therefore

$$H_z = 18.56 \cos\left(2\times 10^{8}\, t - px\right)$$

Example 5

If the magnitude of ***E*** in a plane wave is 377 V/m, determine the magnitude of ***H*** for a plane wave in free space.

Solution: We have

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega \ \Rightarrow H = \frac{377}{377} = 1 \text{ A/m}$$

Example 6

A parallel-plate capacitor with circular plates of radius $a = 0.055$ m is being charged at a uniform rate so that the electric field between the plates changes at a constant rate

$$\frac{\partial \vec{E}}{\partial t} = 1.5\times 10^{13} \text{ V/m/s}$$

Determine the displacement current for the capacitor.

Solution: The displacement current density between the plates of the capacitor is

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Displacement current

$$\vec{I}_D = (\pi a^2)\, \vec{J}_D = \pi a^2 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Here

$$\frac{\partial \vec{E}}{\partial t} = 1.5\times 10^{13} \text{ V/m/s}, \ a = 0.055 \text{ m and } \varepsilon_0 = 8.85\times 10^{-12} \text{ C}^2\text{N-m}^2$$

Displacement current

$$\vec{I}_D = \pi a^2 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = 1.3 \text{ A}$$

Example 7

A lamp radiates 500 W power uniformly in all directions. Calculate the electric and magnetic field intensities at 1 m distance from the lamp.

Solution: As we know Poynting vector $\vec{P} = \vec{E} \times \overrightarrow{H}$ is the energy flowing through unit area and unit time. Now

$$\text{Area} = 4\pi r^2 = 4\pi(1)^2 = 4\pi \text{ m}^2$$

Now

$$\vec{P} = \frac{500}{4\pi} \text{ Joule/m}^2\text{/sec}$$

or

$$EH = \frac{500}{4\pi}$$

But we know that

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377 \ \Omega$$

So $E = 377\ H$. Multiplying both sides by H and using the value of EH we get

$$377 \times H^2 = \frac{500}{4\pi} \Rightarrow H^2 = \frac{500}{4\pi \times 377}$$

$$\Rightarrow H^2 = 0.105$$

$$\Rightarrow H = 0.33 \text{ A-turn/m}$$

Now

$$EH = \frac{500}{4\pi} \Rightarrow E = \frac{500}{4\pi \times 0.33} = 120.63 \text{ V/m}$$

Example 8

Earth receives 2 calories of solar energy per minute per cm^2 as an average over a year for whole surface. What are the amplitudes of average electric and magnetic field radiation?

Solution: The energy received by an electromagnetic power flow is given by

$$\vec{P} = \vec{E} \times \overrightarrow{H}$$

$$\Rightarrow \vec{P} = \frac{2 \times 4.2 \times 10^4}{60} = 1400 \text{ Joule/m}^2\text{/s}$$

Now $P = EH$. So $EH = 1400$. But we know that

$$\frac{E}{H} = \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.77 \approx 377 \ \Omega$$

So

$$E \times \frac{E}{377} = 1400 \Rightarrow E^2 = 527240 \Rightarrow E_{\text{avg}} = 726.1 \text{ A-turn/m}$$

Similarly H_{avg} can be calculated as

$$H_{\text{avg}} = \frac{1400}{E}$$

$$\Rightarrow H_{\text{avg}} = 1.928 \text{ A-turn/m}$$

The amplitudes are calculated using the following expression:

$$E_0 = E_{\text{avg}}\sqrt{2} = 1.414 \times 726.1 = 1026.7 \text{ A-turn/m}$$

Similarly $H_0 = 2.726$ A-turn/m.

Example 9

Calculate the skin depth for a frequency of 10^{20} Hz for silver if $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\sigma = 3 \times 10^7$ S/m.

Solution: We know that

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

Given that $\mu_0 = 4\pi \times 10^{-7}$ Weber/A-m, $\sigma = 3 \times 10^7$ S/m and $\omega = 2\pi f = 10^{20}$. So

$$\delta = \frac{1}{k} = \sqrt{\frac{2}{4\pi \times 10^{-7} \times 3 \times 10^7 \times 2\pi \times 10^{20}}} = 0.091 \times 10^{-10} \text{ m}$$

Short Answers of Some Important Questions

1. How was the idea of electromagnetic waves conceived?

Answer: Faraday's law suggests that a time-varying magnetic field produces an electric field while Ampere's law shows that a time-varying electric field produces a magnetic field. Using this fact, Maxwell showed that if either of the electric or magnetic field changes with time, a field of another kind is induced in the adjacent space and produces waves which are called electromagnetic waves.

2. What do you understand by electromagnetic waves?

Answer: Electromagnetic waves consist of changing electric and magnetic fields. The electric and magnetic components of plane electromagnetic wave are perpendicular to each other and also perpendicular to the direction of the propagation. These waves propagate in space from one position to another even in absence of material medium.

3. Write down some properties of electromagnetic waves.

Answer: The properties of electromagnetic waves travelling through free space are as follows:

1. Electromagnetic waves travel with the speed of light.
2. Electromagnetic waves are transverse waves.
3. The ratio of electric to magnetic field in an electromagnetic wave equals the speed of light.
4. Electromagnetic waves carry both energy and momentum.

4. Give some examples of electromagnetic waves.

Answer: Radio waves, light, X-rays, γ-rays, etc. are the examples of electromagnetic waves.

5. What is displacement current?

Answer: The rate of change of electric displacement vector with time is known as displacement current. In other words, one can say that the displacement current is the current arising due to time-varying electric field between the plates of the capacitor.

6. What is the role of displacement current in electromagnetics?

Answer: On the basis of displacement current, the symmetry character of electric field and magnetic field is more prominent. With the introduction of current density, a changing electric field is now seen to produce magnetic field just as a changing magnetic field gives rise to electric field. Thus, higher degree of symmetry of electric and magnetic field is more satisfactory. Also on the basis of displacement current, both steady and non-steady current circuits may be analyzed as well as all the variations in AC circuits with a capacitor can be easily understood.

7. Differentiate between conduction current and displacement current.

Answer:

1. Conduction current is due to the actual flow of current in a conductor while displacement current is the result of time-varying electric field in a dielectric.
2. Conduction current density is the product of electrical conductivity and electric field; however, displacement current density is the rate of change of electric displacement vector with time.
3. Conduction current obeys Ohm's law while displacement does not obey Ohm's law.

8. Write down Maxwell's equations in dielectric media.

Answer: In dielectric medium there is no free charge. So $\sigma = 0$, $J = 0$ and $\rho = 0$. Therefore, Maxwell's equations are as follows:

$$\nabla \cdot \vec{D} = 0\,; \ \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,; \ \nabla \times \overrightarrow{H} = \frac{\partial \vec{D}}{\partial t}$$

9. Show that $\vec{E}/\vec{B} = c$, where c is the velocity of electromagnetic wave.

Answer: We know that

$$\vec{k} \times \vec{E} = w\vec{B}$$

Since we have already discussed that electric field vector is perpendicular to the direction of propagation, so

$$\vec{k} \cdot \vec{E} = w\vec{B}$$

or $$\frac{\vec{E}}{\vec{B}} = \frac{w}{\vec{k}} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda$$

or $$\frac{\vec{E}}{\vec{B}} = c \ [\because c = \nu\lambda]$$

10. What is Poynting vector?

Answer: $\vec{P} = \vec{E} \times \overrightarrow{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector. It is also called the flux vector. The SI unit of Poynting vector is Wm^{-2}.

Important Points and Formulas

1. With the change in electric and magnetic field with time, a field of other kind is induced in the adjacent space which produces electromagnetic waves consisting electric and magnetic fields.
2. Maxwell formulated the concept of displacement current to remove the inconsistency in Ampere's law by adding the term $\vec{J}_D = \frac{\partial \vec{D}}{\partial t}$.
3. The current arising due to time-varying electric field between the plates of a capacitor is called the displacement current.
$$\vec{J}_d = \varepsilon_0 \frac{dE}{dt}$$
4. The equation of continuity is based on the conservation of charge.
5. Electromagnetic waves propagate with the speed of light
$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$
6. The speed of light in a material is always less than in vacuum because ε_r has a value greater than one.
7. According to Poynting theorem, the rate at which electromagnetic energy in a finite volume decreases with time is equal to the rate of dissipation of energy in the form of joule heat plus the rate at which energy flows out of the volume.
8. $\vec{P} = \vec{E} \times \overrightarrow{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector. It is also called the flux vector. The SI unit of Poynting vector is Wm^{-2}.
9. Skin depth is the depth in conducting medium in which the amplitude of the electromagnetic wave is reduced $(1/e)$ times of its original value.
10. The skin depth is frequency dependent for good conductor and frequency independent in poor conductor.

Multiple Choice Questions

1. Displacement current is due to
(a) displacement of electric charges
(b) time varying magnetic field
(c) time varying electric field
(d) Both (b) and (c)

2. Equation of continuity is based on
(a) conservation of charges
(b) conservation of momentum
(c) conservation of angular momentum
(d) None of these

3. Time varying electric field in the region between the plates is equivalent to
(a) conduction current
(b) displacement current
(c) Both (a) and (b)
(d) Neither (a) nor (b)

4. Who observed that a time varying magnetic field gave rise to an electric field.
(a) Maxwell **(b)** Ampere
(c) Oersted **(d)** Faraday

5. Poynting theorem represents
(a) conservation of charges
(b) conservation of momentum
(c) conservation of energy
(d) None of these

6. Maxwell observed and corrected a discrepancy in
(a) Ampere's Law
(b) Faraday's Law
(c) Gauss Law for electrostatics
(d) None of these

7. According to maxwell's equation in free space; $\nabla \cdot E = ?$
(a) ρ **(b)** 0
(c) ρ/ε_o **(d)** $\frac{\varepsilon_0}{\rho}$

8. In a conducting medium, the electromagnetic waves are.
(a) amplified **(b)** attenuated
(c) both (a) & (b) **(d)** None of the these

9. Energy density in electric and magnetic field is
   (a) Different (b) 1.5
   (c) L/C (d) Same
10. The wave velocity in non-conducting medium is
   (a) $\frac{1}{\sqrt{\mu\varepsilon}}$
   (b) $\sqrt{\mu/\varepsilon}$
   (c) $\sqrt{\frac{\mu_0}{\varepsilon_0}}$
   (d) $\frac{1}{\sqrt{\mu_0\varepsilon_0}}$
11. The characteristic impedance of free space is
   (a) 0 (b) 1
   (c) 377 (d) None of these

Short Answer Type Questions

1. What do you understand by electromagnetic waves?
2. What are Maxwell's equations?
3. What do you mean by displacement current?
4. Differentiate between conduction current and displacement current.
5. What is current density?
6. Write down Maxwell's equations for free space.
7. What is Poynting vector?
8. What do you understand by impedance?

Long Answer Type Questions

1. Explain the concept of Maxwell's displacement current and show how it led to the modification of Ampere's law.
2. Derive Maxwell's equations. Explain the physical significance of each equation.
3. Derive the electromagnetic wave equations in vacuum. Hence show that the waves travel at a speed of light.
4. Derive Poynting theorem. Explain each term.
5. Prove that electromagnetic waves propagate with speed of light.

Numerical Problems

1. Determine refractive index and velocity of light if the relative permittivity of distilled water is 81.
2. A uniform plane wave has electric field intensity in air as 7500 V/m in the *y*-direction. The wave is propagating in the *x*-direction at a frequency of 2×10^9 rad/s. Determine the frequency, wavelength, time-period and amplitude of ***H***.
3. If the magnitude of ***E*** in a plane wave is 455 V/m, determine the magnitude of ***H*** for a plane wave in free space.
4. A parallel-plate capacitor with circular plates of radius $a = 0.55$ cm is being charged at a uniform rate so that the electric field between the plates changes at a constant rate $\frac{\partial \vec{E}}{\partial t} = 1.5 \times 10^{13}$ V/m/s. Determine the displacement current for the capacitor.
5. A lamp radiates 400 W power uniformly in all directions. Calculate the electric and magnetic field intensities at 1.5 m distance from the lamp.

Answers

Multiple Choice Questions

1. (c)
2. (a)
3. (b)
4. (d)
5. (c)
6. (a)
7. (b)
8. (b)
9. (d)
10. (a)
11. (c)

Numerical Problems

1. 9, 3.33×10^{7} m/s
2. 3.18×10^{8} Hz, 0.94 m, 3.14×10^{-9} s and 19.89 A/m
3. 1.21 A/m
4. 1.3×10^{-2} A
5. 796.18 V/m and 0.04 A-turn/m

12 Semiconductors

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Band theory of solids.
- Density of states.
- Fermi–Dirac distribution.
- Free carrier density (electrons and holes).
- Conductivity of semiconductors.
- Position of Fermi level in intrinsic and in extrinsic semiconductors.

12.1 Introduction

Semiconductors are materials that have their conductivity between conductor and insulator. The word semiconductor is derived from the combination of ‘semi’ which means half and ‘conductor’ which means metal. Thus, we can say that the semiconductor is a material which behaves sometimes as a metal and at other times as an insulator. The resistivity of semiconductor is greater than that of a conductor but less than that of an insulator. At room temperature, due to thermal energy, some of the electrons in the semiconductor are able to conduct electricity to a small extent. Germanium with resistivity of about 0.6 Ω-m and silicon having a resistivity of 1.5×10^3 Ω-m are examples of semiconductors.

12.2 Types of Semiconductors

There are two types of semiconductors: One is intrinsic and the other is extrinsic semiconductor.

12.2.1 Intrinsic Semiconductors

A pure semiconductor without any impurities is known as intrinsic semiconductor. In intrinsic semiconductor, the electron and hole concentrations are equal because these carriers within a pure material are created in pairs. For example, pure germanium (Ge) or Silicon (Si).

12.2.2 Extrinsic Semiconductors

A doped impure semiconductor is known as extrinsic semiconductor. Doping increases the conductivity of semiconductors. On the basis of doping extrinsic semiconductors are classified as n-type and p-type. The majority carriers in n-type and p-type are electrons and holes, respectively; the minority carriers in n-type and p-type are holes and electrons, respectively. For example, when a small amount of pentavalent impurity like arsenic or phosphorus is added to a pure germanium (Ge) or Silicon (Si), it is known as n-type semiconductor; however, when a small amount of trivalent impurity like gallium or boron is added to a pure germanium (Ge) or Silicon (Si), it is known as p-type semiconductor.

12.3 Band Theory of Solids

On the basis of free electron theory we can explain the phenomena of electrical conductivity, electron emission, etc. But this theory fails to explain why some solids are conductors, some are insulators and some are semiconductors. The free electron theory of metals assumes that potential is constant in a metal; however, the potential is not constant and may vary with the position of electrons in the metal. The actual nature of potential under which an electron moves in a metal is very complicated. Therefore, these phenomena can be explained on the basis of band theory of solids. According to this theory, the potential experienced by an electron passing through a metal is one which is perfectly periodic with period equal to the lattice constant for one-dimensional case.

12.4 Energy Bands in Solids

The electrons in an isolated atom have a well-defined set of energy levels. When two identical atoms are close together their electrons move under the influence of the combined electric fields of the two atoms and each previously single energy level splits into two levels, one higher and one lower than the corresponding level of the isolated atoms. When large numbers of atom are together, as in a crystal, the energy levels spread into bands. Such bands are called energy bands.

12.4.1 Valence Band

The outermost orbit electrons are called valence electrons. An energy band occupied by the valence electron is called valence band. This band may be completely or partially filled.

12.4.2 Conduction Band

The next higher permitted energy band is known as conduction band or the lowest unfilled energy band is called conduction band. At room temperature the valence electrons may become free; these are called conduction electrons. The range of energies possessed by these conduction electrons is known as conduction band.

12.4.3 Forbidden Band

Valence and conduction bands are separated by an energy gap; this gap is known as forbidden gap or forbidden band. In the forbidden band, no electron can stay because there is no allowed energy state. The energy gap for germanium is 0.72 eV and that for silicon is 1.12 eV.

12.5 Conductivity of Semiconductors

Under the action of external electric field, the electrons drift toward the positive cathode and the holes move toward the negative electrode thereby constituting the current. A potential difference developed between both the ends of a sample makes the charge carrier attain the drift velocity v_d. This drift velocity is proportional to applied external field strength E. Thus

$$v_d \propto E \Rightarrow v_d = \mu E \tag{12.1}$$

where μ is the mobility of charge carriers. We know that the current density due to electron is defined as

$$J_n = n\,e\,v_{dn} \tag{12.2}$$

where v_{dn} is the drift velocity of electron and can be obtained using Eq. (12.1) as

$$v_{dn} = \mu_n E \tag{12.3}$$

If σ_n is the conductivity of the semiconductor due to electrons, then the current density due to electrons will be

$$J_n = \sigma_n E$$

$$\Rightarrow \sigma_n = \frac{J_n}{E} \tag{12.4}$$

Substituting the value of J_n from Eq. (12.2) into Eq. (12.4), we have

$$\sigma_n = \frac{nev_{dn}}{E}$$

Using Eq. (12.3) we have

$$\sigma_n = \frac{n\,e\,\mu_n E}{E} \Rightarrow \sigma_n = n\,e\mu_n \tag{12.5}$$

This is the expression for conductivity of semiconducting material due to electrons. Similarly, we can obtain the expression of the conductivity for the hole, which is given by

$$\sigma_p = pe\mu_p \tag{12.6}$$

Thus the total conductivity of the material is

$$\sigma = \sigma_n + \sigma_p = ne\mu_n + pe\mu_p = e(n\mu_n + p\mu_p)$$

But in the case of intrinsic semiconductor $n = p = n_i$. So

$$\sigma = e(\mu_n + \mu_p)\, n_i \tag{12.7}$$

where n_i is the number of electrons or holes per unit volume.

12.6 Density of States

The number of electrons or holes per unit volume in conduction or valence band whose energy lies between E and $E + dE$ is given by

$$n_i = \int Z(E)\, F(E)\, dE \tag{12.8}$$

where $F(E)$ is the Fermi–Dirac distribution function and depends upon the probabilities associated with the distribution of electrons of the system of energy E. $Z(E)$ is the energy density of the states.

The density of states for electrons in a band yields the number of states in certain energy ranges. This function is important in electronic processes, particularly in transport phenomena. Thus, *density of states [Z(E)dE] is defined as number of electron states per unit volume in the energy range E and E + dE.*

To evaluate $Z(E)$, consider n-space with radius n. Therefore, the number of states lying in the cell, that is, in the energy range E and $E + dE$ is

$$Z(E)\,dE = \frac{1}{8V} 4\pi\, n^2 dn$$

that is volume of the octant of the cell divided by volume of a single state (V). In order to take into account the spin degeneracy (degenerate states have a common eigenvalue) we multiply this expression by 2. Hence

$$Z(E)\,dE = 2 \cdot \frac{1}{8V} 4\pi\, n^2 dn \tag{12.9}$$

We know that energy (E) of an electron of mass m in a box of side l in three dimensions is

$$E = \frac{n^2 h^2}{8ml^2} \qquad \text{or} \qquad n^2 = \frac{8ml^2}{h^2} E \tag{12.10}$$

After differentiation we get

$$2n\,dn = \frac{8ml^2}{h^2} dE$$

or

$$dn = \frac{8ml^2}{2nh^2} dE = \frac{8ml^2}{2\left(\frac{8ml^2}{h^2}\right)^{1/2} E\,h^2} dE = \frac{1}{2}\left(\frac{8ml^2}{h^2}\right)^{1/2} E^{-1/2} dE \tag{12.11}$$

Now substituting the values of n^2 and dn from Eqs. (12.10) and (12.11), respectively, in Eq. (12.9) we have

$$Z(E)\,dE = 2 \cdot \frac{1}{8V} 4\pi \frac{8ml^2}{h^2} E \cdot \frac{1}{2}\left(\frac{8ml^2}{h^2}\right)^{1/2} E^{-1/2} dE = 2 \cdot \frac{\pi}{4V}\left(\frac{8ml^2}{h^2}\right)^{3/2} E^{1/2} dE$$

or

$$Z(E)\,dE = \frac{4\pi}{h^3}(2m)^{3/2} E^{1/2} dE \tag{12.12}$$

From Eq. (12.12), it is clear that the function $Z(E)$ increases with E because the larger the energy, the greater the radius, and hence the volume of the shell, and consequently the larger the number of states lying with in it. It is also reflected that the larger the mass, the greater the density of states.

12.7 Fermi–Dirac Distribution

Fermi–Dirac distribution is applicable to electrons in a solid crystal which obeys the restriction imposed by Pauli's Exclusion Principle (that is, no two particles can occupy the same state). According to Fermi, the probability $F(E)$ that a state of energy E is filled is given by

$$F(E) = \frac{1}{1 + e^{(E - E_F)/kT}} \tag{12.13}$$

where E_F is called the Fermi energy level and is defined as the highest energy that an electron can have in a conductor at absolute zero temperature, k is Boltzmann constant and T is absolute temperature.

12.8 Free Carrier Density or Concentration of Electrons in the Conduction Band

In the conduction band, electrons are free to move anywhere as a free electrons, therefore the number of electrons in conduction band per unit volume whose energy lies between E and $E + dE$ is given by

$$n_e = \int_{E_C}^{\infty} Z(E)F(E)dE \tag{12.14}$$

If $Z(E)$ is the energy density of the states at the bottom of the conduction band, then

$$Z(E)\,dE = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}(E - E_C)^{1/2}\,dE \tag{12.15}$$

where E_C is the energy at the bottom of the conduction band. According to Fermi–Dirac distribution, $F(E)$ is given by

$$F(E) = \frac{1}{1 + e^{(E-E_F)/kT}} \tag{12.16}$$

Putting the values from Eqs. (12.15) and (12.16) in Eq. (12.14), we have

$$n_e = \int_{E_C}^{\infty} \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}(E - E_C)^{1/2}\frac{1}{1 + e^{(E-E_F)/kT}}\,dE$$

$$\Rightarrow n_e = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\int_{E_C}^{\infty}(E - E_C)^{1/2}\frac{1}{1 + e^{(E-E_F)/kT}}\,dE \tag{12.17}$$

If $E - E_F >> kT$ or $(E - E_F)/kT >> 1$, then the term i in the denominator is negligible. So the above equation will become

$$n_e = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\int_{E_C}^{\infty}\frac{(E - E_C)^{1/2}}{e^{(E-E_F)/kT}}\,dE = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\,(E - E_C)^{1/2}\int_{E_C}^{\infty} e^{-(E-E_F)/kT}dE$$

$$= \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\,(E - E_C)^{1/2}\int_{E_C}^{\infty} e^{-(E - E_C + E_C - E_F)/kT}dE$$

$$= \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}(E - E_C)^{1/2}e^{(E_F - E_C)/kT}\int_{E_C}^{\infty} e^{(E_C - E)/kT}dE \tag{12.18}$$

Let

$$\frac{E - E_C}{k\,T} = P$$

$$\Rightarrow \frac{dE}{kT} = dP \Rightarrow dE = kTdP \Rightarrow (E - E_C)^{1/2} = P^{1/2}(kT)^{1/2}$$

When $E = \infty$ we have $x = \infty$ and if $E = E_C$ then $x = 0$.
Now

$$n_e = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\ e^{(E_F - E_C)/k\,T}\int_0^\infty e^{-P} kTP^{1/2}(kT)^{1/2}\,dP$$

$$= \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2}\ e^{(E_F - E_C)/k\,T}\int_0^\infty e^{-P} P^{1/2}(kT)^{3/2}\,dP$$

$$= \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2} e^{(E_F - E_C)/kT}\,(kT)^{3/2}\int_0^\infty e^{-P} P^{1/2} dP$$

After integration we get

$$\int_0^\infty e^{-P} P^{1/2} dP = \frac{\sqrt{\pi}}{2}$$

Hence

$$n_e = \left(\frac{4\pi}{h^3}\right)(2m_e)^{3/2} e^{(E_F - E_C)/k\,T}\,(kT)^{3/2}\,\frac{\sqrt{\pi}}{2}$$

$$\Rightarrow n_e = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{(E_F - E_C)/k\,T} \tag{12.19}$$

This is the required expression for the carrier density or concentration of electrons in conduction band.

12.9 Free Carrier Density or Concentration of Holes in the Valence Band

In order to find the value of n_h we shall use the term $[1 - F(E)]$ instead of $F(E)$. Therefore

$$1 - F(E) = 1 - \frac{1}{1 + e^{(E-E_F)/k\,T}} = 1 - [1 + e^{(E-E_F)/k\,T}]^{-1}$$

$$\Rightarrow 1 - F(E) = \ 1 - [1 - e^{(E-E_F)/k\,T}]$$

$$\Rightarrow 1 - F(E) = 1 - [1 - e^{(E-E_F)/k\,T}]$$

$$\Rightarrow 1 - F(E) = e^{(E-E_F)/k\,T}$$

The value of $Z(E)$ is given by

$$Z(E)dE = \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\left(E_V - E\right)^{1/2} dE \tag{12.20}$$

where E_V is the energy at the top of the valence band. Then the value of n_h is

$$n_h = \int_{-\infty}^{E_V} Z\,(E)\,(1 - F\,(E))\,dE$$

$$n_h = \int_{-\infty_C}^{E_V} \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\,(E_V - E)^{1/2}\,e^{(E-E_F)/kT}\;dE$$

$$= \int_{-\infty_C}^{E_V} \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\,(E_V - E)^{1/2}\,e^{(E_V - E_F + E - E_V)/kT}\,dE$$

$$= \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\,e^{\frac{E_V - E_F}{kT}} \int_{-\infty_C}^{E_V} (E_V - E)^{1/2}\,e^{(E - E_V)/kT}\,dE$$

Let

$$\frac{E_V - E}{kT} = P$$

Then

$$-dE = kTdP \Rightarrow (E_V - E)^{1/2} = P^{1/2}(kT)^{1/2}$$

When $E = -\infty$ then $x = \infty$ and if $E = E_V$ then $x = 0$. So

$$n_h = \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\;e^{\frac{E_V - E_F}{kT}} \int_{\infty}^{0} e^{-P}\cdot(-kT)\cdot P^{1/2}\cdot(kT)^{1/2}\;dP$$

$$n_h = \left(\frac{4\pi}{h^3}\right)(2m_h)^{3/2}\;e^{\frac{E_V - E_F}{kT}} \int_{0}^{\infty} e^{-P}\cdot P^{1/2}\cdot(kT)^{3/2}\;dP <$$

After integration we get

$$\int_0^{\infty} e^{-P} P^{1/2} dP = \frac{\sqrt{\pi}}{2}$$

Hence

$$n_h = 2\left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{\frac{E_V - E_F}{kT}} \tag{12.21}$$

This is the required expression for the carrier density or concentration of holes in valence band.

12.10 Position of Fermi Level in Intrinsic and Extrinsic Semiconductors

Fermi energy is the highest energy of the electron in the valence band of a crystal in its ground state. Fermi level is used as a reference level. Let us study how the idea of Fermi level classifies a semiconductor.

We know that in a semiconductor when all the electrons are present in the valence band then the probability

$$F(E) = 1$$

In this state the conduction band is vacant and the probability of the conduction band will be

$$F(E) = 0$$

Therefore, Fermi level having $F(E) = 1/2$ must be somewhere in between these two bands. In an **intrinsic semiconductor** the position of the Fermi level E_F is midway in the forbidden energy gap as shown in Fig. 1.

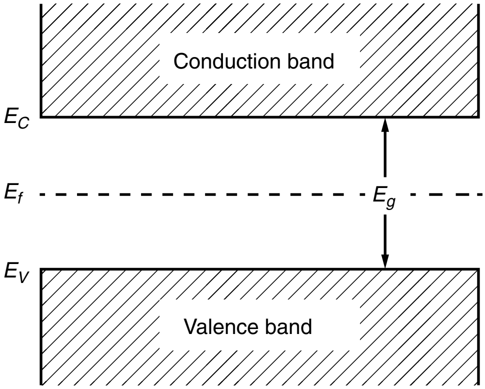


Figure 1 Fermi level in an intrinsic semiconductor.

When we dope pure semiconductor with donor impurities like phosphorous, arsenic, etc. then Fermi level E_F rises above the mean level and when we add acceptor impurities like aluminium, boron, etc., the Fermi level falls below the mean level in the forbidden energy gap.

Thus, in an extrinsic p-type semiconductor, the Fermi level is nearer the valence band while in an extrinsic n-type semiconductor, it is nearer the conduction band as shown in Figs. 2 and 3, respectively.

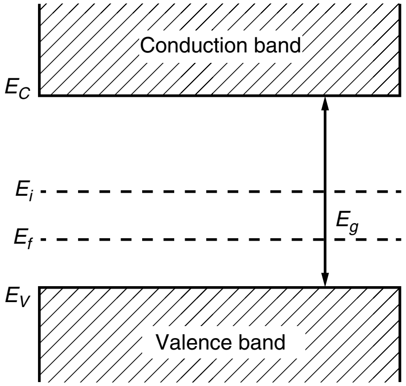


Figure 2 Fermi level in p-type extrinsic semiconductor.

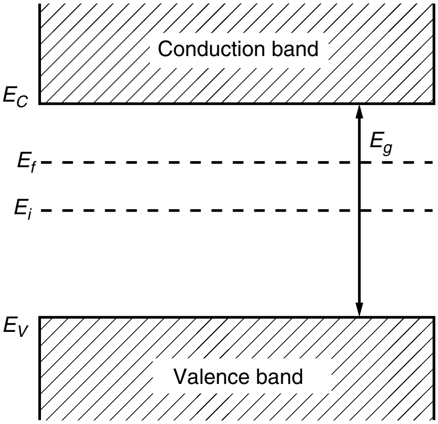


Figure 3 Fermi level in an n-type extrinsic semiconductor.

Solved Examples

Example 1

In a sample of intrinsic germanium at room temperature, the mobility of electrons and holes is 0.40 m^2/V-sec and 0.23 m^2/V-sec, respectively. If the electron and hole densities are each equal to 1.5×10^{23} m^{-3}, find out the electrical conductivity and resistivity of germanium.

Solution: Here we have $\mu_n = 0.40$ m^2/V-sec, $\mu_p = 0.23$ m^2/V-sec. Now $n = p = 1.5\ \times 10^{23}$ m^{-3}. The conductivity of intrinsic semiconductor is given by

$$\sigma = e\,(\mu_n + \mu_p)n_i = 1.6 \times 10^{-19} \times (0.40 + 0.23) \times 1.5\ \times 10^{23} = 1.512 \times 10^4\ \Omega\,\mathrm{m}^{-1}$$

Resistivity of intrinsic semiconductor is given by

$$\rho = \frac{1}{\sigma} = \frac{1}{1.512 \times 10^4} = 0.66 \times 10^{-4}\,\Omega^{-1}\mathrm{m}$$

Example 2

Find the temperature at which there is 1.0% probability that a state with energy 0.5 eV above Fermi energy will be occupied.

Solution: Acording to Fermi distribution law

$$F(E) = \frac{1}{1 + e^{\frac{E - E_F}{KT}}}$$

We have

$$F(E) = 1.0\% = 0.01$$
$$E - E_F = 0.5 \text{ eV} = 0.5 \times 1.6 \times 10^{-19}\,\text{J} = 8 \times 10^{-20}\,\text{J}$$
$$k = 1.38 \times 10^{-23} \text{ J/K}$$

Substituting these in the above formula we get

$$0.01 = \frac{1}{1 + e^{\frac{8\times10^{-20}}{1.38\times10^{-23}T}}} \Rightarrow 1 + e^{\frac{5797.1}{T}} = 100 \Rightarrow \frac{5797.1}{T} = 4.595 \Rightarrow T = 1261.6\,\text{K}$$

Example 3

Find the probility with which an energy level 0.02 eV below Fermi level will be occupied at room temperature of 300 K and at 1000 K.

Solution: Acording to Fermi distribution law

$$F(E) = \frac{1}{1 + e^{\frac{E-E_F}{KT}}}$$

Now we have

$$E - E_F = -0.02 \text{ eV}$$
$$k = 1.38 \times 10^{-23} \text{ J/K} = 8.625 \times 10^{-5} \text{ eV/K}$$

For $T = 300$ K,

$$F(E) = \frac{1}{1 + e^{\frac{-0.02}{8.625\times10^{-5}\times300}}} = 0.684$$

For $T = 1000$ K,

$$F(E) = \frac{1}{1 + e^{\frac{-0.02}{8.625\times10^{-5}\times1000}}} = 0.56$$

Example 4

Calculate the probability of an electron occupying an energy level 0.02 eV above the Fermi level at 200 K and 400 K in a material.

Solution: Acording to Fermi distribution law

$$F(E) = \frac{1}{1 + e^{\frac{E-E_F}{KT}}}$$

We have

$$E - E_F = 0.02 \text{ eV}$$

$$k = 1.38 \times 10^{-23} \text{ J/K} = 8.625 \times 10^{-5} \text{ eV/K}$$

For $T = 200$ K

$$F(E) = \frac{1}{1 + e^{\frac{0.02}{8.625 \times 10^{-5} \times 200}}} = 0.24$$

For $T = 400$ K

$$F(E) = \frac{1}{1 + e^{\frac{0.02}{8.625 \times 10^{-5} \times 200}}} = 0.34$$

Example 5

For an n-type semiconductor with energy gap $E_g = 0.7$ eV, calculate the concentration of n-type charge carriers at 300 K.

Solution: The n-type charge carrier concentration is

$$n_e = 2\left(\frac{2\pi m_e kT}{h^2}\right) e^{E_g/kT}$$

We have

$$m_e = 9.1 \times 10^{-31} \text{ kg}, \; k = 1.38 \times 10^{-23} \text{ J/K}, \; T = 300 \text{ K},$$

$$h = 6.63 \times 10^{-34} \text{ Js}, \; E_g = 0.7 \text{ eV} = 0.7 \times 1.6 \times 10^{-19} \text{ J}$$

So

$$n_e = 2\left(\frac{2\pi 9.1 \times 10^{-31} \times 1.38 \times 10^{-23} \times 300}{(6.6 \times 10^{-34})^2}\right) e^{(-1.12 \times 10^{-19})/(2 \times 1.38 \times 10^{-23} \times 300)} = 3.34 \times 10^{19} /\text{m}^3$$

Example 6

The energy gap of silicon is 1.1 eV. Its electron and hole mobilities at room temperature are 0.45 and 0.043 $\mathrm{m^2V^{-1}}s^{-1.}$ Evaluate the carrier concentration and its electrical conductivity.

Solution: The charge carrier concentration is

$$n_e = 2\left(\frac{2\pi m_e kT}{h^2}\right)e^{E_g/kT}$$

We have

$$m_e = 9.1\times10^{-31}\ \text{kg},\ k = 1.38\times10^{-23}\ \text{J/K},\ T = 300\ \text{K},\ h = 6.63\times10^{-34}\ \text{Js}$$

$$E_g = 1.1\ \text{eV} = 1.1\times1.6\times10^{-19}\ \text{J} = 1.76\times10^{-19}\ \text{J}$$

So

$$n_i = 2\left(\frac{2\times3.14\times9.1\times10^{-31}\times1.38\times10^{-23}\times300}{(6.6\times10^{-34})^2}\right)e^{(-1.76\times10^{-19})/(2\times1.38\times10^{-23}\times300)} = 1.467\times10^{16}/\text{m}^3$$

$$\sigma = n_i e(\mu_e + \mu_h) = 1.467\times10^{16}\times1.6\times10^{-19}\times(0.45+0.043) = 1.16\times10^{-3}\,\Omega^{-1}\text{m}^{-1}$$

Example 7

Calculate the conductivity of pure silicon at room temperature when the concentration of carriers is 1.6×10^{10} per cm^3. Given that $\mu_e = 1300\ \text{cm}^2/\text{V-s}$ and $700\ \text{cm}^2/\text{V-s}$.

Solution: We know that

$$\sigma_i = n_i e(\mu_e + \mu_h)$$

We have

$$n_i = 1.6\times10^{10}/\text{cm}^3,\ e = 1.6\times10^{-19}\,\text{C},\ \mu_e = 1300\ \text{cm}^2/\text{V-s},\ \mu_h = 700\ \text{cm}^2/\text{V-s}$$

So

$$\sigma_i = 1.6\times10^{10}\times1.6\times10^{-19}(1300+700) = 5.12\times10^{-6}\ \text{mho/cm}$$

Example 8

Calculate the number of donor atoms which must be added to an intrinsic semiconductor to obtains the resistivity as 10^{-6} ohm cm. Assume $\mu_e = 100 \text{ cm}^2/\text{V-sec}$.

Solution: We know that

$$\rho = \frac{1}{\sigma} = \frac{1}{n_e e \mu_e}$$

Now $\rho = 10^{-6}\ \Omega\text{-cm}$, $e = 1.6 \times 10^{-19}\,\text{C}$, $\mu_e = 100 \text{ cm}^2/\text{V-sec}$ and

$$n_e = \frac{1}{10^{-6} \times 1.6 \times 10^{-19} \times 100} = 6.25 \times 10^{22}/\text{cm}^3$$

Short Answers of Some Important Questions

1. What are semiconductors?

Answer: Semiconductors are those materials which have their conductivity between conductor and insulator.

2. What is intrinsic semiconductor?

Answer: A pure semiconductor without impurities is known as intrinsic semiconductor. In intrinsic semiconductor, the electron and hole concentrations are equal because these carriers within a pure material are created in pairs.

3. What do you meant by energy bands?

Answer: When large numbers of atoms are together, as in a crystal, the energy levels spread in to bands. Such bands are called energy bands.

Important Points and Formulas

1. Semiconductors are materials that have their conductivity between conductor and insulator.
2. The resistivity of semiconductor is greater than that of a conductor but less than that of an insulator.
3. A pure semiconductor without any impurities is known as intrinsic semiconductor.
4. The outermost orbit electrons are called valence electrons.
5. The next higher permitted energy band is known as conduction band.
6. Valence and conduction bands are separated by an energy gap; this gap is known as forbidden gap or forbidden band. In the forbidden band, no electron can stay because there is no allowed energy state. The energy gap for germanium is 0.72 eV and that for silicon is 1.12 eV.
7. The total conductivity of the material is

$$\sigma = \sigma_n + \sigma_p = ne\mu_n + pe\mu_p = e(n\mu_n + p\mu_p)$$

But in the case of intrinsic semiconductor $n = p = n_i$. So

$$\sigma = e(\mu_n + \mu_p)\, n_i$$

where n_i is the number of electrons or holes per unit volume.

8. In an **intrinsic semiconductor** the position of the Fermi level E_F is midway in the forbidden energy gap.

9. In an extrinsic p-type semiconductor, the Fermi level is nearer the valence band while in an extrinsic n-type semiconductor, it is nearer the conduction band.

Multiple Choice Questions

1. The forbidden energy gap in an insulator is
(a) 6 eV **(b)** 3 eV
(c) 0 eV **(d)** 1 eV

2. The forbidden energy gap in a semiconductor is
(a) 6 eV **(b)** 3 eV
(c) 0 eV **(d)** 1 eV

3. The resistivity of conductors
(a) increases with temperature
(b) decreases with temperature
(c) is independent of temperature
(d) None of these

4. The resistivity of insulator
(a) is low **(b)** is high
(c) remains constant **(d)** None of these

5. Conductivity of conductors near room temperature
(a) increases with increasing temperature
(b) decreases with increasing temperature
(c) remains constant
(d) first increases and then decreases

6. Conductivity of semiconductors near room temperature
(a) increases with increasing temperature
(b) decreases with increasing temperature
(c) remains constant
(d) first increases and then decreases

7. Position of Fermi level in intrinsic semiconductor is
(a) above the valence bond
(b) below the conduction band
(c) midway of the valence and conduction band
(d) None of these

8. In p-type semiconductor the Fermi level lies
(a) above the valence bond
(b) below the conduction band
(c) midway of the valence and conduction band
(d) None of these

9. In n-type semiconductor the Fermi level lies
(a) above the valence bond
(b) below the conduction band
(c) midway of the valence and conduction band
(d) None of these

10. For n-type semiconductors
(a) $n_e >> n_h$
(b) $n_e = n_h$
(c) $n_e << n_h$
(d) None of these

11. For p-type semiconductors
(a) $n_e >> n_h$
(b) $n_e = n_h$
(c) $n_e << n_h$
(d) None of these

Short Answer Type Questions

1. What do you understand by a semiconductor?

2. Differentiate between intrinsic and extrinsic semiconductors with examples.

3. Differentiate between conductor, semiconductor and insulator with suitable examples.

4. Explain band theory of solids.

5. What do you understand by energy bands?
6. Explain conduction and valence band.
7. What is Fermi level?
8. What do you understand by conductivity in semiconductor?

Long Answer Type Questions

1. Prove that the density or concentration of electrons in the conduction band of an intrinsic semiconductor is

$$n_e = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{(E_F - E_C)/kT}$$

2. Prove that the density or concentration of electrons in the valence band is

$$n_h = 2\left(\frac{2\pi m_h kT}{h^2}\right)^{3/2} e^{\frac{E_V - E_F}{kT}}$$

3. Explain Fermi–Dirac distribution.
4. Explain with the help of diagrams, the effect of doping on Fermi level of semiconductors.
5. Differentiate between intrinsic and extrinsic semiconductors on the basis of Fermi level.
6. What is Fermi level? Why does a pure semiconductor behave like an insulator at absolute zero?
7. What do you understand by conductivity in semiconductor?
8. Derive an expression for total conductivity in a semiconductor.
9. What is Fermi level? Explain Fermi level in n-type semiconductor.

Numerical Problems

1. In a sample of intrinsic germanium at room temperature, the mobility of electrons and holes is 0.50 m^2/V-sec and 0.13 m^2/V-sec, respectively. If the electron and hole densities are each equal to 2.5×10^{23} m^{-3}, find out the electrical conductivity and resistivity of germanium.
2. Calculate the number of donor atoms which must be added to an intrinsic semiconductor to obtains the resistivity as 10^{-6} ohm cm. Assume $\mu_e = 1000$ cm^2/V-sec.
3. Calculate the conductivity of pure silicon at room tmperature when the concentration of carriers is 1.6×10^{10} per cm^3. Given that $\mu_e = 1500$ cm^2/V-s and 500 cm^2/V-s.
4. The energy gap of silicon is 1.1 eV. Its electron and hole mobilities at room temperature are 0.48 and 0.013 m^{2}V^{-1}s^{-1}. Evaluate the carrier concentration and its electrical conductivity.
5. For a p-type semiconductor with energy gap E_g = 0.7 eV, calculate the concentration of p-type charge carriers at 300 K.

Answers

Multiple Choice Questions

1. (c)
2. (d)
3. (a)
4. (b)
5. (a)
6. (b)
7. (c)
8. (a)
9. (b)
10. (a)
11. (c)

Numerical Problems

1. 2.52×10^{4} mho/cm and $0.396 \times 10^{-4}\ \Omega^{-m}$
2. 6.25×10^{21} percm^{3}
3. 5.12×10^{-6} mho/cm
4. $1.16 \times 10^{-3}\ \Omega^{-1}\text{m}^{-1}$
5. $3.34 \times 10^{19}/\text{m}^{3}$

13 Superconductivity

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Temperature dependence of resistivity in superconducting materials.
- Effect of magnetic field (Meissner effect).
- Temperature dependence of critical field.
- Type I and Type II Superconductors.
- BCS theory (Qualitative).
- High temperature superconductors.
- Applications of superconductors.

13.1 Introduction

Superconductivity is a phenomenon of disappearance of electrical resistance in some materials when cooled to a very low temperature. There are certain materials or alloys whose electrical resistivity suddenly drops to zero (or conductivity increases to infinite) when they are cooled to a very low temperature nearly absolute zero (often called liquid helium temperature range); these materials/alloys are called superconductors. The temperature at which the resistivity becomes zero or material becomes superconductor is called transition temperature or critical temperature. This phenomenon was discovered by Dutch physicist Heike Kamerlingh Onnes[1] on April 8, 1911 in Leiden. He observed the resistance of Hg nearly zero at temperature 4.2 K. Like ferromagnetism, nanoscience, nanotechnology and atomic spectral lines, superconductivity is better explained by quantum mechanical phenomenon. It is characterized by the Meissner effect, the complete expulsion of magnetic field lines from the interior of the material as it transitions into the superconducting state. The occurrence of the Meissner effect indicates that superconductivity cannot be understood simply as the idealization of perfect conductivity in classical physics.

Thus, when a substance is cooled below a certain temperature (called transition temperature or critical temperature), the electrical resistance of the substance suddenly drops to zero; this phenomenon is known as superconductivity and the substances which exhibit this property are known as superconductors.

OR

The phenomenon of disappearance of electrical resistance of material below a certain temperature is called superconductivity and the material is said to be superconductor.

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[1] Onnes was honored in 1913 with Noble prize in physics for his research of matter at low temperatures.

13.2 Temperature Dependence of Resistivity in Superconductors

During the study of electrical resistance of mercury at very low temperature, Onnes in 1911 observed that the electrical resistance of a pure mercury at very low temperatures[2] decreases regularly like that of any other metal but at temperature about 4.2 K, it suddenly drops to zero as shown in Fig. 1. *The temperature at which the normal material changes into superconducting state is called the critical temperature* T_c. Other substances which exhibit this phenomenon are aluminium, silver, cadmium, lead, gallium, iridium, etc. There are so many other alloys and ceramics that behave as superconductors. It may be possible that individual elements may not be the superconductor but their alloy can be superconductor. Figure 1 shows the comparison of resistivity of normal metal and superconductor. From this figure, it is clear that the critical temperature separates the superconducting state from the normal state. Above critical temperature T_c the specimen is in the normal state with finite resistivity but below T_c, it changes into superconducting state with infinite conductivity. This critical temperature and its range are different for different materials and are affected very much by the impurity in the material. A small amount of impurity can change from steep fall of resistivity to gradual fall. Further, this transition of material is reversible and the material transforms to normal state when temperature rises above critical temperature.

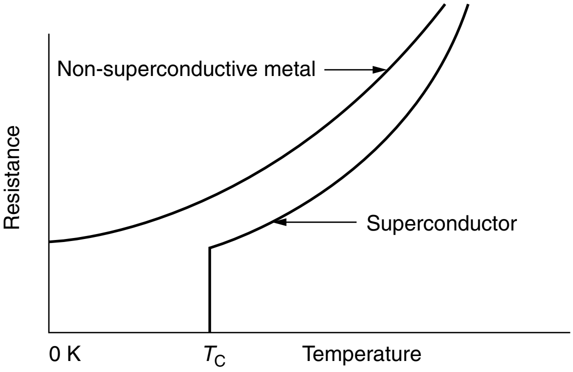


Figure 1 Schematic representation of the electrical resistance of the normal metal and superconductor as a function of temperature.

13.3 Critical Field

When the superconducting materials are subjected to a strong magnetic field, it will result in the destruction of the superconducting property below the critical temperature, that is, they return to the normal state. *Thus, the minimum field required to destroy the superconducting property is called the critical field* H_c. Its value depends upon the nature of the material and its temperature as shown in Fig. 2. The variation of H_c with temperature is nearly parabolic and can be expressed by the equation

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

---

[2] The study of the laws of perfect gases tells us that the lowest possible temperature on the centigrade scale is 273° below the melting point of ice. This temperature (–273°C) is taken as the zero of the absolute temperature scale. Thus, very low temperature means the region from 0°C to absolute zero (–273°C).

where $H_c(0)$ is the critical field at 0 K and $H_c(T)$ is the critical field at temperature T. Below critical temperature, the critical field decreases with increase of temperature. It is maximum at absolute zero and zero at critical temperature. It means that no field is required for transition from superconductor to normal state at critical temperature.

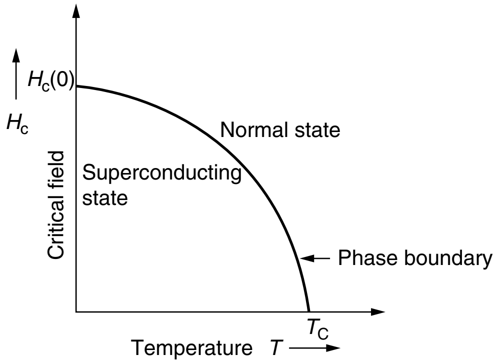


Figure 2 Schematic representation of the critical field as a function of temperature.

13.4 Critical Current and Current Density

The application of a large value of electric current to a superconducting material destroys its superconducting property. The current required for this is called critical current and the corresponding current density is called critical current density.

If a superconducting wire of radius r carries current I then critical current I_c is given by

$$I_c = 2\pi r H_c$$

At $I = I_c$ superconductivity will be destroyed. In presence of applied transverse magnetic field H, the value of critical current decreases and expressed as

$$I_c = 2\pi r(H_c - 2H)$$

This is called Silsbee's rule.

Current Density

$$J_c = \frac{I_c}{A} = \frac{I_c}{\pi r^2} \text{ A/m}^2$$

13.5 Effect of Magnetic Field (Meissner Effect)

The Meissner effect is an expulsion of a magnetic field from a material during its transition to the superconducting state. German physicists Walther Meissner and Robert Ochsenfeld discovered the phenomenon in 1933 by measuring the magnetic field distribution outside superconducting tin and lead specimen. The specimens, in the presence of an applied magnetic field, were cooled to a temperature below the

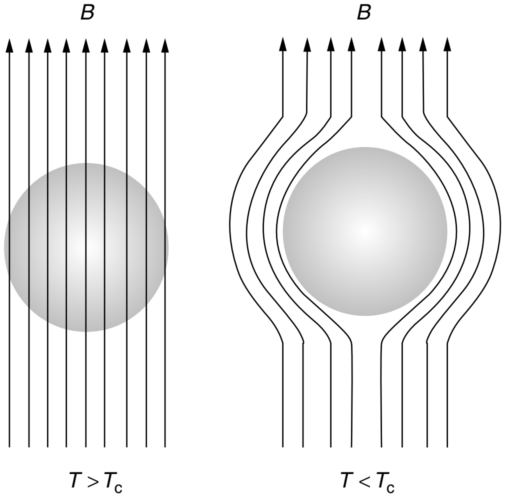


Figure 3 Diagram of the Meissner effect. Magnetic field lines, represented as arrows, are excluded from a superconductor when it is cooled below critical temperature.

superconducting transition temperature. Until the transition temperature, the magnetic lines of force penetrate the interior of specimen; however as the temperature decreases below the transition temperature, the magnetic flux is expelled out from the interior and goes round of the specimen as shown in Fig. 3. In fact, in the presence of magnetic field below the critical temperature, currents on the surface of the material (persistent current) are generated which actually cancel the interior flux density of the superconductor. That is to say, below the transition temperature the material becomes **perfectly diamagnetic**. *Thus, when a superconductor is placed in an external magnetic field and cooled below the critical temperature, all the magnetic flux is expelled or pushed out of it; this is called Meissner's effect.*

or

The phenomenon of an expulsion or pushing out of magnetic flux when it is cooled below the transition temperature is called Meissner's effect.

The Meissner's effect is of fundamental importance because it shows that in an external magnetic field, a metal in a superconducting state behaves as a perfect diamagnetic. If inside the specimen $B = 0$, then the relation between magnetizing field H, magnetic induction B and intensity of magnetization I is expressed as

$$B = \mu_0(H + I) = 0$$

or

$$\frac{I}{H} = -1$$

According to the definition of susceptibility

$$\chi_{\mathrm{m}} = \frac{I}{H} = -1$$

This shows that the susceptibility of a superconductor is −1 (negative) and the relative permeability is zero. This is one of the characteristics of a perfect diamagnetic.

We know that resistivity of a superconductor becomes zero; therefore, electric field E must be zero for finite current density. Now from Maxwell's third equation, we have

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

Since $E = 0$

$$\frac{\partial B}{\partial t} = 0 \quad \text{or} \quad B = \text{constant}$$

This concludes that the magnetic flux in the interior of the superconductor is independent of the transition temperature. This contradicts Meissner effect. Hence, the resistivity and perfect diamagnetism are two independent and essential properties of a superconductor.

13.6 Type I and Type II Superconductor

Superconducting materials are classified as Type I (soft) and Type II (hard) superconductors depending on their characteristic behaviors in the presence of an external magnetic field. Type I superconductors exhibit complete Meissner effect, that is below critical field H_c they are superconductors and at H_c they become normal as shown in Fig. 4(a). The value of H_c is very small and is of the order of 0.1 T. This type of superconductivity is normally exhibited by pure metals, for example, aluminium, lead, indium and mercury.

On the other hand, Type II superconductors exhibit two critical magnetic fields H_{c1} and H_{c2} as materials behave in different ways [see Fig. 4(b)]. Below critical field H_{c1}, the magnetization increases linearly, the material is superconductor and exhibits complete Meissner effect that is complete expulsion of magnetic line of forces from the bulk material. Above H_{c1}, the magnetization decreases gradually and magnetic lines start penetrating the material. The material loses its superconducting state and comes to normal sate at H_{c2}. The region between H_{c1} and H_{c2} is called mix state. If we compare with Type I superconductor, there is abrupt fall in magnetization at A wherein Type II fall is gradual. Transition metals and alloys consisting of niobium and aluminium are examples of Type II superconductors.

A material can change from Type I to Type II on addition of some impurities. For example, lead is a Type I superconductor with $H_c = 4.8 \times 10^4$ A/m at 4°K and $H_{c2} = 7.96 \times 10^4$ A/m. On adding 20% of indium in lead, $H_{c1} = 5.6 \times 10^3$ A/m and $H_{c2} = 2.9 \times 10^5$ A/m. Magnetization curves for both Type I and Type II materials are reversible in case of the ideal superconductors.

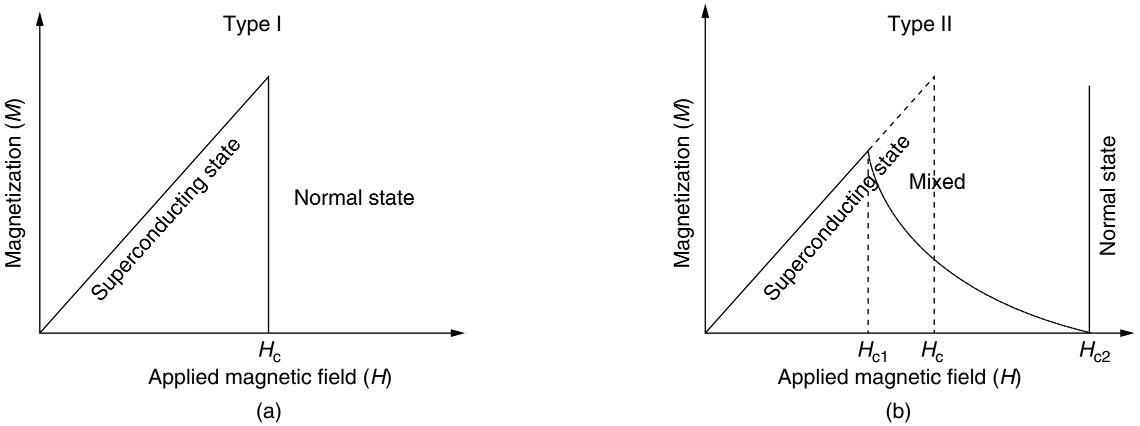


Figure 4 Magnetization versus magnetic field for (a) Type I and (b) Type II superconductors.

13.7 BCS Theory

John Bardeen, Leon N. Cooper and John R. Schrieffer (BCS) developed the quantum theory of superconductivity, the ability of certain metals at low temperatures to conduct electricity without electrical resistance. It is based on interaction of two electrons through the intermediary of phonons. The behavior of *superconductors* suggests that electron pairs are coupling over a range of hundreds of nanometers, there orders of magnitude larger than the lattice spacing. These pairs are called **Cooper pairs**. These coupled electrons can take the character of a boson and condense into the ground state as shown in Fig 5. BCS showed that the basis of interaction responsible for superconductivity to be a pair of electron is by means of interchange of virtual phonons. Usually the electrons repel each other by the columbic force but in special case when temperature is lower than critical value, the attraction of electron is stronger than electrons columbic force. The attraction force can be explained in following way: Suppose that an electron approaches a positive core, it suffers attractive coulomb interaction. Due to this interaction ion core is set in motion and consequently distorts the lattice. Smaller the mass of the positive ion core, greater will be the distortion. Let another electron come towards that and see this distorted lattice. Then the interaction between distorted lattice and electron takes place which affects the lowering energy of the second electron. Thus, the second electron interacts via lattice distortion or the phonon field resulting in the lowering of energy of the electrons. This lowering energy of electron implies that the force between the two electrons is attractive. This lowering of interaction is called electron-lattice-electron interaction. It is strongest when the two electrons have equal and opposite spin and momenta. This is the state where Cooper pair of electrons smoothly ride over the lattice imperfections without exchanging energy with them. Hence no transfer of energy takes place from the Cooper pair to the lattice ion.

Alternately, we can say that a pair of free electrons coupled through a phonon is called a Cooper pair. The energy of Cooper pair is lower than that of the individual electrons. Since electrons with opposite spin can become paired, therefore, the total spin of pair is zero. As a result the electrons pairs in a superconductor are bosons.

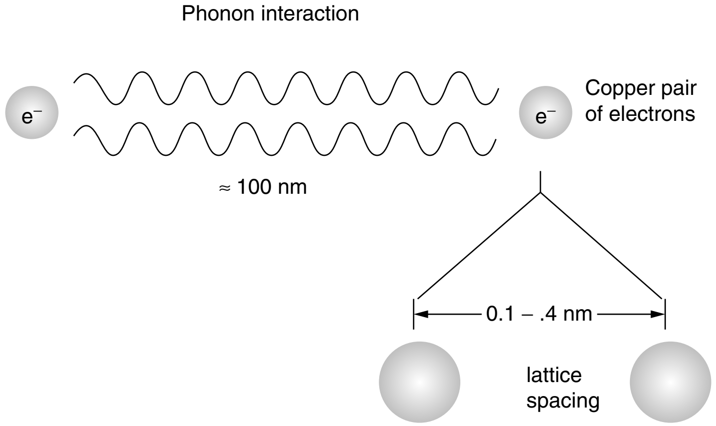


Figure 5 Schematic of cooper pair.

13.8 High-Temperature Superconductivity

Until 1986, physicists had believed that BCS theory forbade superconductivity at temperatures above about 30 K. In that year, Bednorz and Müller discovered superconductivity in a lanthanum-based cuprate-perovskite material, which had a transition temperature of 35 K (Nobel Prize in Physics, 1987). It was a great achievement for discoverer to have the superconductors above 10 K transition temperature and the new nomenclature of high temperature superconductor was introduced. It was soon found that replacing lanthanum with yttrium (i.e., making YBCO) raised the critical temperature to 92 K. This temperature jump is particularly significant, since it allows liquid nitrogen as a refrigerant, replacing liquid helium. This can be commercially important because liquid nitrogen can be produced relatively cheap, even on-site, avoiding some of the problems (such as so-called 'solid air' plugs) which arise when liquid helium is used in piping. Many other cuprate superconductors have since been discovered, and the theory of superconductivity in these materials is one of the major outstanding challenges of theoretical condensed matter physics. Since about 1993, the highest temperature superconductor was a ceramic material consisting of mercury, thallium, barium, calcium, copper, and oxygen ($Hg_{12}Tl_3Ba_{30}Ca_{30}Cu_{45}O_{125}$) with $T_c = 133 - 138$ K. The latter experiment (138 K) still awaits experimental confirmation. In February 2008, an iron-based family of high-temperature superconductors was discovered. However, the room temperature superconductor are yet to be discovered.

13.9 Characteristics of Superconductors

The following are the important characteristics of superconductors:

1. The current in superconductors can be sustained for a long period.
2. Magnetic field does not penetrate the superconductor.
3. Above critical magnetic field, a superconductor becomes normal metal.
4. The materials having high resistivity are superconductors.
5. Ferromagnetic and anti-ferromagnetic materials are not superconductors.

13.10 Applications of Superconductors

Superconducting magnets are some of the most powerful electromagnets. They are used in MRI/NMR machines, mass spectrometers, etc. The beam-steering magnets are used in particle accelerators. They can also be used for magnetic separation, where weakly magnetic particles are extracted from a background of less or non-magnetic particles, as in the pigment industries.

In the 1950s and 1960s, superconductors were used to build experimental digital computers using cryotron switches. More recently, superconductors have been used to make digital circuits based on rapid single flux quantum technology and RF and microwave filters for mobile phone base stations.

Superconductors are used to build Josephson junctions which are the building blocks of SQUIDs (superconducting quantum interference devices), the most sensitive magnetometers known. SQUIDs are used in scanning SQUID microscopes and magnetoencephalography. Series of Josephson devices are used to realize the SI volt. Depending on the particular mode of operation, a superconductor-insulator-superconductor Josephson junction can be used as a photon detector or as a mixer. The large resistance change at the transition from the normal to the superconducting state is used to build thermometers in cryogenic micro-calorimeter photon detectors. The same effect is used in ultrasensitive bolometers made from superconducting materials. Other early markets arise where the relative efficiency, size and weight advantages of devices based on high-temperature superconductivity outweigh the additional costs involved. Promising future applications include high-performance smart grid, electric power transmission transformers, power storage devices, electric motors (e.g. for vehicle propulsion, as in vactrains or maglevtrains), magnetic levitation devices, fault current limiters, nanoscopic materials such as buckyballs, nanotubes, composite materials and superconducting magnetic refrigeration. However, superconductivity is sensitive to moving magnetic fields, so that applications that use alternating current (e.g. transformers) will be more difficult to develop than those that rely upon direct current.

Solved Examples

Example 1

The critical field for Al is 1.2×10^4 A/m. Determine the critical current and current density which can flow through a long thin superconducting wire of aluminium of diameter 1 mm.

Solution: Here we have diameter = 1 mm = 10^{-3} m, $H_c = 1.2 \times 10^4$ A/m, radius = 0.5×10^{-3} m. Now

$$I_c = 2\pi r H_c = 2 \times 3.14 \times 0.5 \times 10^{-3} \times 1.2 \times 10^4 = 37.68\ A$$

Current density

$$J_c = \frac{I_c}{\pi r^2} = \frac{37.68}{3.14 \times (0.5 \times 10^{-4})^2}\ \text{A/m}^2 = 48 \times 10^8\ \text{A/m}^2$$

Example 2

A superconducting material has a critical temperature of 3.7 K in zero magnetic field and a critical field of 0.02 T at 0 K. Find the critical field of 3 K.

Solution: Here we have $T_c = 3.7$ K, $H_c(0) = 0.02\ T$ and $T = 3$ K. Now

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right] = 0.02\left[1 - \left(\frac{3}{3.7}\right)^2\right] = 0.0069\ \text{T}$$

Example 3

The critical field for lead is 1.2×10^5 A/m at 8 K and 2.4×10^5 A/m at 0 K. Find the critical temperature of the material.

Solution: Here we have $H_c(T) = 1.2 \times 10^5$ A/m, $H_c(0) = 2.4 \times 10^5$A/m and $T = 8$ K. Now

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right] \Rightarrow \frac{H_c(T)}{H_c(0)} = \left[1 - \left(\frac{T}{T_c}\right)^2\right] \Rightarrow \frac{T}{T_c} = \sqrt{1 - \frac{H_c(T)}{H_c(0)}}$$

$$\Rightarrow T_c = \frac{T}{\sqrt{1 - \frac{H_c(T)}{H_c(0)}}} = \frac{8}{\sqrt{1 - \frac{1.2 \times 10^5}{2.4 \times 10^5}}} = 11.31\ \text{K}$$

Example 4

The critical fields for lead are 1.2×10^5 A/m and 3.6×10^5 A/m at 12 K and 10 K, respectively. Find its critical temperature and critical field at 0 K and 3.2 K.

Solution: Here we have

$$H_c(T) = 1.2 \times 10^5 \text{ A/m at } T = 12 \text{ K}$$

$$H_c(T) = 3.6 \times 10^5 \text{A/m at } T = 10 \text{ K}$$

Now

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

So we have

$$\frac{1.2 \times 10^5}{H_c(0)} = \left[1 - \left(\frac{12}{T_c}\right)^2\right] \tag{13.1}$$

$$\frac{3.6 \times 10^5}{H_c(0)} = \left[1 - \left(\frac{10}{T_c}\right)^2\right] \tag{13.2}$$

Dividing Eq. (13.1) by Eq. (13.2), we get

$$1 - \frac{144}{T_c^2} = 0.33 - \frac{33}{T_c^2} \;\Rightarrow\; T_c^2 = 165.67 \;\Rightarrow\; T_c = 12.87$$

Now substituting the value of T_c in Eq. (13.1) we get

$$\frac{1.2 \times 10^5}{H_c(0)} = \left[1 - \left(\frac{12}{12.87}\right)^2\right] \;\Rightarrow\; H_c(0) = 9.12 \times 10^5 \text{ A/m}$$

Critical field is given by

$$H_c(T) = 9.12 \times 10^5\left[1 - \left(\frac{3.2}{12.87}\right)^2\right] = 8.56 \times 10^5 \text{ A/m}$$

Example 5

Determine the temperature at which the critical field becomes half of its value at 0 K if critical temperature of a superconductor when no magnetic field is present is T_c.

Solution: Here we have

$$H_c(T) = \frac{H_c(0)}{2}$$

Now using this formula in

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

we have

$$\frac{H_c(0)}{2} = H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

$$\Rightarrow T = \frac{T_c}{\sqrt{2}} = 0.71\,T_c$$

Short Answers of Some Important Questions

1. What is superconductivity?

   Answer: The phenomenon of disappearance of electrical resistance of material below a certain temperature is called superconductivity.
2. What is a superconductor?

   Answer: The material in which electrical resistance of material disappears below a certain temperature is said to be superconductor.
3. What do you mean by critical field?

   Answer: The minimum field required to destroy the superconducting property is called the critical field.
4. What do you understand by critical current and critical current density?

   Answer: The current required to destroy the superconducting property of a superconducting material is called critical current and corresponding current density is called critical current density.
5. What is the effect of magnetic field on a superconductor?

   Answer: When a superconductor is placed in an external magnetic field and cooled below the critical temperature, all the magnetic flux is expelled or pushed out of it. It is also known as Meissner's effect.
6. What is the importance of Meissner's effect?

   Answer: Meissner's effect is of fundamental importance because it shows that in an external magnetic field, a metal in a superconducting state behaves as perfect diamagnetic.
7. What are the essential properties of a superconductor?

   Answer: Zero resistivity and perfect diamagnetism are the two independent and essential properties of a superconductor.
8. What is Cooper pair?

   Answer: The behavior of *superconductors* suggests that electron pairs couple over a range of hundreds of nanometers, there orders of magnitude larger than the lattice spacing; such pairs are called *Cooper pairs*.

Important Points and Formulas

1. The phenomenon of disappearance of electrical resistance of material below a certain temperature is called superconductivity and the material is said to be superconductor.
2. The temperature at which the normal material changes into superconducting state is called the critical temperature T_c.
3. The minimum field required to destroy the superconducting property is called the critical field H_c.
4. The variation of H_c with temperature is nearly parabolic and can be expressed by the equation

$$H_c(T) = H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

   where $H_c(0)$ is thecritical field at 0 K and $H_c(T)$ is the critical field at temperature T.

5. The resistivity and perfect diamagnetism are two independent and essential properties of a superconductor.

6. A pair of free electrons coupled through a phonon is called a Cooper pair. The energy of Cooper pair is lower than that of the individual electrons. Since electrons with opposite spin can become paired, therefore, the total spin of pair is zero. As a result the electrons pairs in a superconductor are bosons.

7. If a superconducting wire of radius r carries current I then critical current I_c is given by

$$I_c = 2\pi\, rH_c$$

8. In presence of applied transverse magnetic field H, the value of critical current decreases and expressed as

$$I_c = 2\pi\, r(H_c - 2H)$$

This is called Silsbee's rule.

Multiple Choice Questions

1. A superconductor is
(a) paramagnet **(b)** diamagnet
(c) Both (a) and (b) **(d)** None of these

2. The energy gap in a superconductor
(a) increases with increasing temperature
(b) is independent of temperature
(c) decreases with increasing temperature
(d) None of these

3. The electrical resistance or resistivity of a pure superconductor at transition temperature
(a) gradually reaches zero
(b) suddenly becomes zero
(c) is independent of temperature
(d) None of these

4. The transition temperature of a superconductor
(a) depends on the isotropic mass
(b) is independent of mass
(c) does not depend on charge
(d) None of these

5. The number of critical field in a Type II superconductor is
(a) 0 **(b)** 1
(c) 2 **(d)** 4

6. Meissner effect takes place in
(a) liquid
(b) superconducting magnet
(c) MRI
(d) Maglev vehicle

Short Answer Type Questions

1. What do you understand by a superconductor?
2. What is Meissner's effect?
3. What do you understand by soft and hard superconductors?
4. Give some properties of superconductors.
5. What is critical current?
6. What do you understand by transition temperature?
7. Explain Type I superconductor.
8. What do you mean by high temperature superconductor?

Long Answer Type Questions

1. What do you understand by superconductivity? Discuss temperature dependence of superconductor.
2. What are superconductors? Describe the effect of magnetic field on superconductors.
3. Explain Meissner's effect of superconductivity.

4. What is critical current? Explain Silsbee's effect in superconductors.
5. What do you understand by critical current density?
6. What is critical field? Distinguish between soft and hard superconductors.
7. What is Meissner effect? Prove that Meissner effect and the disappearance of resistivity in a superconductor are mutually consistent.
8. Write a short note on Type I and Type II superconductors.
9. What are superconductors? Discuss applications of superconductor.
10. What is BCS theory?
11. Explain the behavior of a superconductor in magnetic field.
12. Write a short note on high temperature superconductor.

Numerical Problems

1. The critical field for iridium is 3.4×10^4 A/m. Determine the critical current and current density which can flow through a long thin superconducting wire of iridium of radius 1 mm.
2. A superconducting material has a critical temperature of 4.8 K in zero magnetic field and a critical field of 3.4×10^4 A/m at 0 K. Find the critical field of 3.5 K.
3. A lead wire has a critical magnetic field of 5.5×10^4 A/m at 0 K. The critical temperature is 6.2 K. At what temperature would the critical field drop to 3.5×10^4 A/m?
4. The critical magnetic field for lead wire is 5.5×10^4 A/m. What are the critical current and critical current density if the diameter of the wire is 1 mm?
5. A lead wire has a critical magnetic field of 5.5×10^4 A/m at 0 K. The critical temperature is 6.2 K. At what temperature would the critical field drop to 3.5×10^4 A/m? What is the critical current density at that temperature if the diameter of the wire is 1 mm?
6. The critical fields for lead is 1.8×10^6 A/m at 6 K and 2.4×10^6 A/m at 0 K. Find the critical temperature of the material.
7. The critical fields for lead are 1.2×10^5 A/m and 2.4×10^5 A/m at 14 K and 13 K, respectively. Find its critical temperature and critical field at 0 K and 4.2 K.
8. Determine the temperature at which the critical field becomes one-third of its value at 0 K if critical temperature of a superconductor when no magnetic field is present is T_c.

Answers

Multiple Choice Questions

1. (b) **2.** (a) **3.** (b) **4.** (a) **5.** (c) **6.** (d)

Numerical Problems

1. 197.2 A and 197.2×10^6 A/m^2
2. 1.59×10^2 A/m
3. 3.74 K
4. 172.7 A and 220×10^6 A/m^2
5. 3.74 K, 126×10^6 A/m^2
6. 3 K
7. 10×10^5 A/m and 9.2×10^5 A/m
8. 1.2 T_c

14 Nanotechnology

LEARNING OBJECTIVES

After reading this chapter, you will be able to understand:

- Basic principle of nanoscience and technology.
- Structure, properties and uses of Fullerene and Carbon nanotubes.
- Applications of nanotechnology.

14.1 Introduction

Nanotechnology, often shortened to 'nanotech', is the study of the control of matter on an atomic and molecular scale. Generally, nanotechnology deals with structures of the size 100 nanometers (*one nanometre is* 10^{-9} *m*) or smaller in at least one dimension, and involves developing materials or devices within that size. Nanotechnology is very diverse, encompassing numerous fields in the natural sciences. It is devoted to develop, design, synthesize and apply the structure by manipulating atoms and molecules at nanoscale. Since, Classical or Newtonion mechanics is not applicable to explain the physical properties of materials such as mechanical, optical, electrical, chemical, etc. on moving macroscopic scale to nanoscale, therefore, these characteristics of different materials can be explained directly or indirectly by quantum mechanics.

Now-a-days, nanoscience and nanotechnology are concerned with the production of various materials having dimensions of the order of a billionth of a meter. These materials are known as nanomaterials and produced in nanoscale in one dimension (for example, thin surface coatings), in two dimensions (for example, nanotubes and nanowires) and in three dimensions (for example, nanoparticles and quantum dots). Many other nanostructures such as semiconductor nanowire, nanofiber, nanorod components are synthesized for nanodevices. One aspect of nanomaterials is to increase the surface area to volume ratio altering various properties like mechanical, thermal catalytic properties. It has huge potential to develop new material and objects which can have a wide range of application in different areas and can be beneficial to make significant impact on society, such as healthcare, electronics, information technology and energy storage.

14.1.1 Nanoscience and Nanotechnology

Nanoscience and nanotechnology deal with research and development of matter in the smallest range of scale. Nanoscience is the branch of physics in which we study the phenomenon and manipulation of materials at atomic, molecular and micromolecular scales. Nano is a Greek word that means dwarf (small). On the other hand, **nanotechnology** is a branch in which we study the design, characteristics, production and application of structure, devices and systems on the nanoscale.

The growth of nanoscience and nanotechnology is in line with the tendency towards miniaturization. It is a cutting edge technology where objects or devices behave as a whole unit that has size from 1 to

100 nanometers. As nanoscience and nanotechnology develop, they make the obstacles between traditional scientific and technological disciplines more permeable.

We know that every matter consists of atoms or molecules which exist in solid, liquid and gaseous states. Nanoscience makes the study of solid particles, wires, thin films, etc. having dimensions below than 100 nanometers interesting. The comparative size of nano and meter is same as that of a marble to the earth.

Nanoscience and nanotechnology have become worldwide interdisciplinary fields for research and development. Our traditional disciplines such as material science, applied physics, applied chemistry, chemical engineering, mechanical engineering, electrical engineering, food engineering, biological engineering, etc. have become more on an application of nanotechnology. This technology is to understand and think of about the synthesis, properties and applications of material from existing scale to nanoscale.

The materials developed under nanotechnology show very different properties at nanoscale in comparison to macroscale though the properties hardly change at microscale. For instance, opaque substances at macroscale become transparent at nanoscale (Cu). Materials having inert properties attain catalytic properties (Pt), stable materials turn into combustible materials (Al), solids turn into liquids (Au), insulators become conductors (Si), etc. Nanomaterials in powder forms (known as nanoparticles) are potentially important in ceramics, powder metallurgy, the achievement of uniform nanoporosity and similar applications.The differences in properties of same materials is due to their structure and bonding. So nanotechnology is the understanding of relationship between physical properties and nanoscale dimensions.

14.1.2 Nanoparticles

Nanoparticles are atomic clusters with grain size in nanometer range. Nanoparticles or nanocrystals (which are made of metal, semiconductors or oxides) are of great scientific interest for their mechanical, electrical, magnetic, opticle, chemical and other properties. Nanoparticles are effectively a bridge between macroscale and nanoscale. For example, Cu nanoparticles smaller than 50 nm are considered superhard materials while at 50 nm or above, movement occurs in the atoms/cluster (*cluster is nothing but a group of metal atoms*) of Cu.

Experimental evidences show that there are two types of structure in nanomaterials: one is geometrical structure and the other is electronic structure. In geometrical structure, possible arrangements of atoms for the cluster are as follows:

1. Face centered cubic (fcc)
2. Hexagonal closed pack (hcp)
3. Icosahedral structure (icos)

The geometrical structure of nanoparticles depends on size. The larger nanoparticles have fcc structure (e.g., aluminium nanoparticles of size 80 nm have fcc structure) and when the size becomes less than a critical size, deviations occur due to the formation of nanoparticles. On the other hand, electronic structure of nanoparticles depends on the size of the particle. If the size of particles is reached when the surfaces of the particles are separated by distances of the order of de-Broglie wavelengths associated with electrons, then the energy levels are to be modified by quantum mechanically. This spilitting of energy levels is known as quantum size effect. Nanoparticles are used as quantum dots and as chemical catalyst. Since the nanoparticles can enter the cells and tissues, therefore these are also medically and environmentally dangerous.

14.2 Nanomaterials

Nanomaterials are materials which are developed, designed, fabricated or manipulated at nanometer scale. As discussed, the various physical, electrical, mechanical, magnetic properties of material at this scale change drastically. These are fabricated from carbon, ceramic, ferrites, metals, polymers, semiconductors and silicas. For example, buckyballs, the structure of C_{60} atoms, nanotubes, nanogels, nanocrystals, Quantum dots, etc.

14.2.1 Properties of Nanomaterials

There are some useful properties of nanomaterials which are as follows:

1. Nanomaterials are hard.
2. Nanomaterials are exceptionally strong.
3. Nanomaterials are ductile at high temperature.
4. Nanomaterials are chemically very active.
5. Nanomaterials are wear resistant.

14.3 Types of Nanomaterials

There are two types of nanomaterials: (a) Fullerenes and (b) Nanoparticles. We will discuss them in the following subsections.

14.3.1 Fullerenes

Fullerenes are a class of allotropes of carbon. Generally, graphene sheets are rolled into tubes or spheres. It is a one-atom thick layer of graphite. A fullerene commonly refers to a molecule with 60 carbon atoms, C_{60}, and with an icosahedral symmetry; however, it also includes larger molecular weight fullerenes like C_{70}, C_{76}, C_{78}, C_{80}, and higher mass fullerenes, which possess different geometrical structure. For example, C_{60} has icosahedral symmetry, C_{70} has a rugby ball-shaped symmetry and C_{80} molecule has icosahedron shape. Figure 1 shows the structure and geometry of some fullerene molecules. The name ‘Fullerene’ was given to this family of carbon molecules because of the resemblance of these molecules to the geodesic dome designed and built by the American architect R. Buckminister Fuller, whereas the name of buckminister fullerene or buckyball was especially given to the C_{60} molecules, which are the most widely studied in the fullerene family because it is the smallest member of fullerenes family.

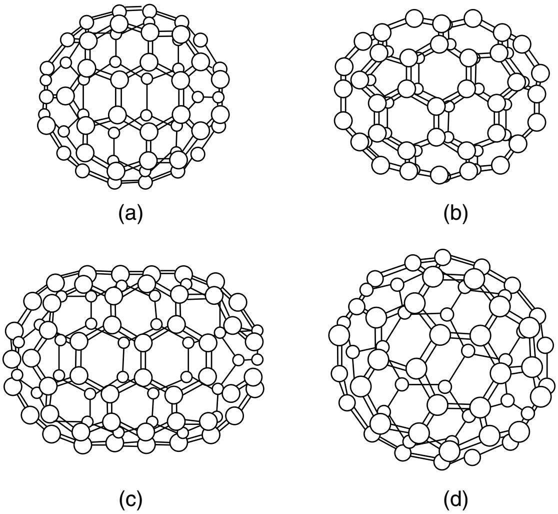


Figure 1 (a) Icosahedral C_{60} molecule. (b) Rugby ball-shaped C_{70} molecule. (c) Extended rugby ball-shaped C_{80} molecule. (d) Icosahedron C_{80} molecule.

Basically, buckyball is a hollow cluster of C_{60} atoms and looks like a football. The 60 carbon atoms in C_{60} are located at the vertices of a regular truncated icosahedral and every carbon site on C_{60} is equivalent to the other site. Each carbon atom is trigonally bonded. Buckyball is the roundest and the most symmetrical molecule known in the world. In buckyball, there are 32 faces, 12 of which are pentagonal and 20 hexagonal which have a molecule diameter of 7.10 Å as shown in Fig. 2.

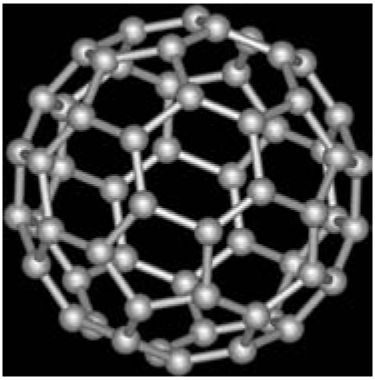


Figure 2 Structure of C_{60} atoms in fullerene or buckyball.

Properties of Buckyball

1. Due to hollow structure, buckyball holds other atoms inside it to create new molecules.
2. They have rigid structure.
3. It can also be used in various applications.
4. They are extremely stable and can withstand at very high temperature and pressure.

Uses of Buckyballs

1. Buckyballs are used for making medicine as well as diagnostic tools.
2. Buckyballs are also used for making anti-wrinkle cream.
3. Buckyballs are used in electronic circuits.
4. Buckyballs are used for making light detectors.

14.3.2 Carbon Nanotubes (CNTs)

Carbon nanotubes are allotropes of carbon with a nanostructure or, in other words, we can say that it is a one-dimensional wire or a seamless cylinder with diameter of the order of a nanometre made by rolling a sheet of graphene. CNTs were discovered by S. Lijima in 1991 and named them nanotubes (see Fig. 3). The name of CNTs was given on the basis of its size. Carbon nanotubes are also the members of fullerenes family.

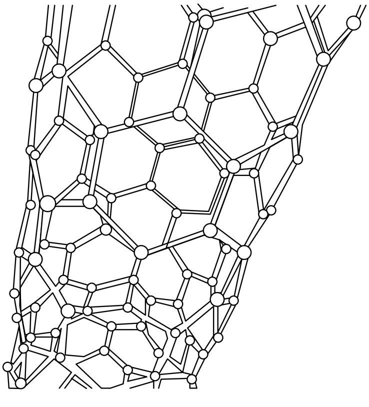


Figure 3 Carbon nanotube.

14.3.2.1 Types of Nanotubes

Carbon nanotubes are divided into two categories:

1. **Single-walled carbon nanotubes (SWCNTs):** Single-walled carbon nanotubes (SWCNTs) can be formed by rolling of a one-atom thick layer of graphene into a seamless cylinder. The diameter of single-walled carbon nanotubes ranges from 1 to 2 nanometer [see Fig. 4(a)]. SWCNT exhibit important electrical properties as they are excellent electric conductors and therefore utilized in running electrical network. The development of first intramolecular field effect transistor (FET) is one of the useful application of single-walled carbon nanotube.
2. **Multi-walled carbon nanotubes (MWCNTs):** Multi-walled carbon nanotubes (MWCNTs) consist of several nested coaxial single wall tubes. In other words, we can say that MWCNTs consist of multiple layers of graphite (called graphene) rolled in on themselves to form a tube shape as shown in Fig. 4. The distance between their walls is about 0.36 nanometer. It is most common and can be easily formed. The diameter of multi-walled carbon nanotubes ranges from 2 to 25 nanometer [see Fig. 4(b)].

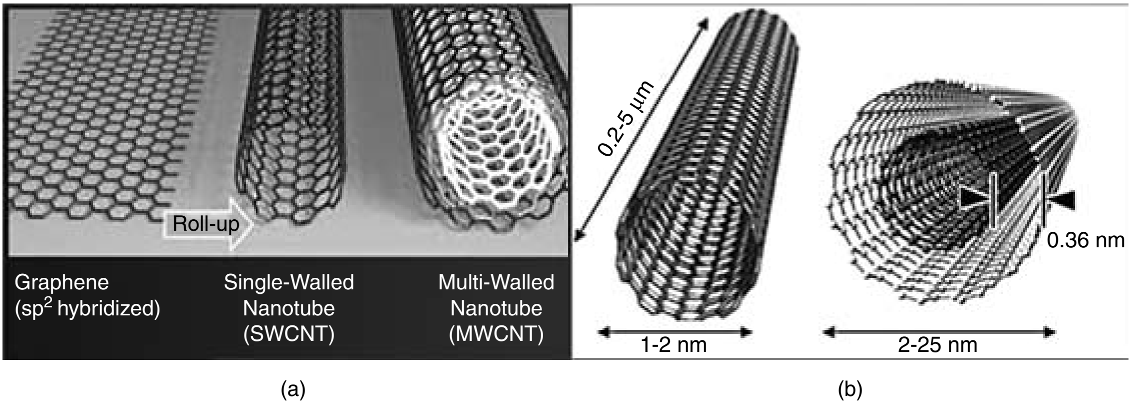


Figure 4 (a) Single-walled carbon nanotube; (b) multi-walled carbon nanotube.

14.3.2.2 Structure of Carbon Nanotubes

The structure of carbon nanotubes is cylindrical based on the hexagonal lattice of carbon atoms that form crystalline graphite. As the graphene sheets can be rolled in different ways, this leads to three types of nanotubes, called **armchair, zig-zag** and **chiral** nanotubes as shown in Fig. 5. From this figure, one can observe that it is possible to recognize armchair, zig-zag and chiral nanotubes by just following the pattern across the diameter of the tubes and analyzing their cross-sectional structure.

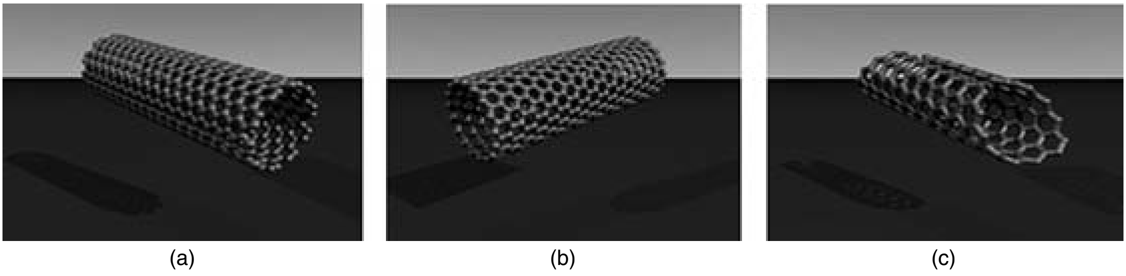


Figure 5 (a) Armchair, (b) zig-zag and (c) chiral nanotubes.

14.3.2.3 Synthesis of Carbon Nanotubes

Various different techniques have been used to produce nanotubes in sizeable quantities. These are arc discharge; laser ablation, high pressure carbon monoxide (HiPCO), and chemical vapour deposition (CVD). We will discuss them in the following subsections.

14.3.2.3.1 Arc Discharge From this method nanotubes were observed in 1991 in the carbon soot of graphite electrodes during an arc discharge. In this process, the carbon contained in the negative electrode evaporates because of the high temperature caused by the discharge. Since nanotubes were initially discovered using this technique, therefore this has been the widely used method of nanotube synthesis. This method is used to produce both single and multi-walled nanotubes with lengths up to 50 micrometres.

14.3.2.3.2 Laser Ablation In this process, a pulsed laser vaporizes a graphite target in a high temperature reactor while an inert gas is bled into chamber. The nanotubes develop on the cooler surfaces of the reactor, as the vaporized carbon condenses. A water cooled surface may be included in the system to collect the nanotubes. This method can produce about 70% primarily SWCNT with controlled diameter determined by reaction temperature. However, this process is more expensive than arc discharge method.

14.3.2.3.3 Chemical Vapour Deposition (CVD) Chemical vapour deposition (CVD) is a common method for the commercial production of carbon nanotubes. In this method, two gases – a process gas such as ammonia, nitrogen, hydrogen and a carbon-containing gas such as methane – are abled into chamber. The carbon-containing gas is broken into carbon atoms at surface of the catalyst particle such as iron, and carbon is deposited to the edges of the iron particle forming nanotubes. The catalyst particles can stay at the trips of the growing nanotubes during the growth process or remain at the nanotube base depending on the adhesion between the catalyst particle and the substrate.

14.3.2.4 Properties of Carbon Nanotubes

Carbon nanotubes have some useful properties as described below:

1. Carbon nanotubes are metallic.
2. Carbon nanotubes are the strongest materials on earth in terms of tensile strength.
3. Carbon nanotubes are elastic.
4. Carbon nanotubes are very good thermal conductors.
5. Carbon nanotubes are light weight.

14.3.2.5 Uses of Carbon Nanotubes

Carbon nanotubes can be used in various applications:

1. They are used for making transistors and memory devices about a nanometer wide.
2. They are used to made nanotubes wire which can conduct huge amount of current with less power wastage.
3. They are also used to develop nanoscale electric motors.
4. They are used in various medical applications.

Short Answers of Some Important Questions

1. What do you understand by nanoscience?

   Answer: Nanoscience is the branch of physics in which we study the phenomenon and manipulation of materials at atomic, molecular and micromolecular scales.

2. What is nanotechnology?

   Answer: Nanotechnology is a branch in which we study the design, characteristics, production and application of structure, devices and systems on the nanoscale.

3. What do you meant by 'nano'?

   Answer: Nano is a Greek word that means dwarf (small).

4. What do you understand by fullerenes?

   Answer: Fullerenes are a class of allotropes of carbon. Generally, graphene sheets are rolled into tubes or spheres. It is a one-atom thick layer of graphite. A fullerene commonly refers to a molecule with 60 carbon atoms, C_{60} and with an icosahedral symmetry, but also includes larger molecular weight fullerenes like C_{70}, C_{76}, C_{78}, C_{80}, and higher mass fullerenes, which possess different geometrical structures.

5. What is the structure of carbon nanotube?

   Answer: The structure of carbon nanotubes is cylindrical based on the hexagonal lattice of carbon atoms that form crystalline graphite.

6. How many types of nanotubes exist?

   Answer: There are three types of nanotubes: armchair, zig-zag and chiral.

Important Points and Formulas

1. Nanotechnology deals with structures of the size 100 nanometers.
2. **Nanoscience** is the branch of physics in which we study the phenomenon and manipulation of materials at atomic, molecular and micro-molecular scales. Nano is a Greek word that means dwarf (small).
3. **Nanotechnology** is a branch in which we study the design, characteristics, production and application of structure, devices and systems on the nanoscale. Nanotechnology is researched and technology development at 1–100 nm range.
4. There are two types of nanomaterials: (a) Fullerenes and (b) nanoparticles.
5. Fullerenes are a class of allotropes of carbon. Generally, graphene sheets are rolled into tubes or spheres.

Multiple Choice Questions

1. Nanophase materials are in the range
   (a) 1–100 nm **(b)** 100–200 nm
   (c) 1000 nm **(d)** None of these
2. The properties of nanoparticles can be explain by
   (a) classical mechanics
   (b) quantum mechanics
   (c) statistical mechanics
   (d) None of these
3. Fullerene is
   (a) molecule
   (b) chemical compound
   (c) Both (a) and (b)
   (d) None of these
4. Carbon nanotubes are
   (a) graphite sheet **(b)** plastic
   (c) graphene sheet **(d)** None of these

5. Which of the following is carbon nanotube?
(a) Armchair
(b) Arch discharge
(c) Chemical discharge
(d) None of these

6. The surface to volume ratio is
(a) low
(b) high
(c) 1
(d) None of these

7. Which of the following forms of pure carbon is similar to soccer ball?
(a) Fullerene **(b)** Diamond
(c) Graphite **(d)** None of these

8. The known third form of the carbon is
(a) fullerene **(b)** diamond
(c) graphite **(d)** None of these

9. Buckyball is the cluster of
(a) 30 carbon atoms **(b)** 60 carbon atoms
(c) 90 carbon atoms **(d)** None of these

Short Answer Type Questions

1. What do you understand by nanomaterials?
2. What are fullerenes?
3. What is the structure of buckyballs?
4. What is a carbon nanotube?
5. What is the property of carbon nanotubes?

Long Answer Type Questions

1. What do you understand by nanoscience and nanotechnology?
2. What are nanomaterials? How can they be fabricated?
3. What are fullerenes? Discuss their properties and applications.
4. Explain the structure of buckyballs.
5. What are carbon nanotubes? Explain their different types.
6. Explain the structure of carbon nanotubes.
7. Explain the various properties of nanotubes.
8. Write some applications of carbon nanotubes.

Answers

Multiple Choice Questions

1. (a)
2. (b)
3. (a)
4. (c)
5. (a)
6. (b)
7. (a)
8. (a)
9. (a)

Engineering Physics Practical

Experiment 1

Date: ____________

Objective

To determine the wavelength of monochromatic (sodium) light by Newton's ring apparatus.

Apparatus

Traveling microscope, support for glass plate inclined at 45°, plano-convex lens, thin glass sheet, sodium lamp and spherometer.

Theory and Formula Used

If the convex surface of a plano-convex lens is placed in contact with a plane glass surface, a thin air film will be formed in between them (Fig. 1). The thickness of the air film increases as one proceeds from the point of contact toward the periphery of the lens.

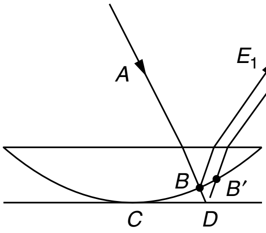


Figure 1

When a monochromatic beam of light is incident on plane surface of such lens, the light is partially reflected from the upper and partially from the lower surfaces of the air film as formed between the lens and the glass plate. Thus, a path difference is introduced between the two reflected rays and consequently the interference fringes are produced. *As the loci of points of equal thickness of the air films are concentric circles with the point of contact as center, the alternate dark and bright fringes will be concentric rings with their center at contact point C.*

Suppose the plano-convex lens touches the glass plate at the point E and B_1, B_2 lie on the nth bright ring whose radius is r_n (Fig. 2). Let R be the radius of curvature of the curved surface of the lens. Then from geometry

$$PB_1 \times PB_2 = XP \times PE$$
$$\Rightarrow (PB_1)_2 = (XE - PE) \times PE$$

$$\Rightarrow (r_n^2) = (2R - t)t \quad [\text{since}\, PB_1 = PB_2 = r \text{ and } XP = 2R]$$

$$\Rightarrow t = \frac{r_n^2}{2R} \quad [\text{since}\, R \gg t] \tag{1}$$

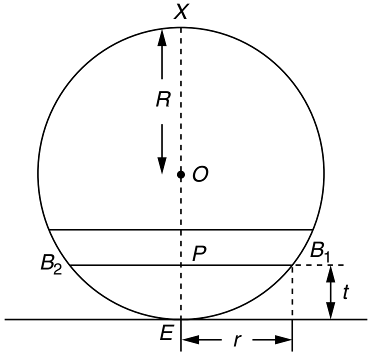


Figure 2

With this thickness of the film, bright ring will occur at the point if

$$2\mu t \cos(\alpha + \theta) = (2n+1)\frac{\lambda}{2} \tag{2}$$

where μ is the refractive index of the film, α is the angle of wedge film, θ is the angle of incidence of light on film and n is any integer.

For air film, $\mu = 1, \theta = 0$ (for normal incidence of the rays) and $\alpha = 0$ for a lens of large radius of curvature. Hence, Eq. (2) becomes

$$t = (2n+1)\frac{\lambda}{4} \tag{3}$$

By Eqs. (1) and (3)

$$r_n^2 = (2n+1)\frac{\lambda R}{2} \tag{4}$$

$$d_n^2 = 2(2n+1)\lambda R \qquad (d_n = \text{diameter of the } n\text{th bright ring}) \tag{5}$$

Similarly for $(n+m)$th bright ring

$$d_{n+m}^2 = 2[2(n+m)+1]\lambda R \tag{6}$$

$$\Rightarrow \lambda = \frac{d_{n+m}^2 - d_n^2}{4Rm} \tag{7}$$

Equation (7) remains the same for dark rings.

Method

Place the plano-convex lens on glass plate in the provided box and place it in such away that sodium light reflected from the inclined glass plate falls normally on the plano-convex lens. The reflected light from both the surfaces of air thin film is received by a traveling microscope (see Fig. 3).

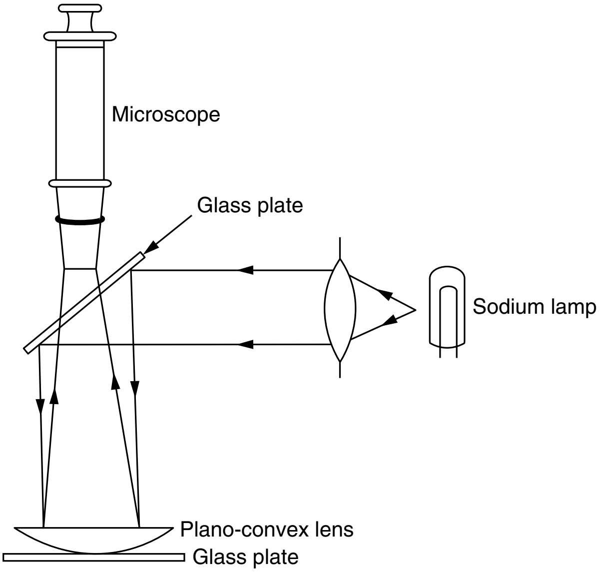


Figure 3

Adjust the microscope over the point of contact so that the rings are seen distinctly in the field of view as clear as possible. Set the cross wire on the center of the rings. Using vernier screw, now move the cross wires and set on say 15th bright ring on any side and read the vernier scale reading. Bring the cross wires of microscope back toward the center, stopping on each ring for vernier readings. Continue to move on other side of center and take similar reading on other side in increasing order of rings until you reach the 15th ring again. Measure R using spherometer (generally given).

The diameters (d) of the rings can be tabulated from the table by taking difference of left side and right side readings of same numbered ring. Square the value of each diameter and then take the difference of these square values of the diameters with some common interval of number m (say $m = 5$) of these bright rings. Now take mean of all such values to finally have value of $d_{n+m}^2 - d_n^2$.

Observations

Radius of curvature of plano-convex lens = ________ cm

Least count of microscope = ____________ cm $\left(\text{LC} = \dfrac{\text{Smallest division of main scale}}{\text{Total no. of divisions on Vernier scale}}\right)$

No. of Rings	***Microscope Reading***		***Diameter in cm***	***d^2 in cm***	$\left(d^2_{n+m} - d^2_n\right)$ $m = 5$	***Mean Value***
	Left Side in cm	***Right side in cm***				
15					$d^2_{15} - d^2_{10} =$	
14						
13					$d^2_{14} - d^2_{9} =$	
12						
11					$d^2_{13} - d^2_{8} =$	
10						
9					$d^2_{12} - d^2_{7} =$	
8						
7					$d^2_{11} - d^2_{6} =$	
6						

Calculation

The wavelength of light can be calculated using formula $\lambda = \dfrac{d^2_{n+m} - d^2_n}{4Rm}$

Conclusion

The wavelength of sodium light was found to be ____________ Å.

Possible Errors

1. The error in setting the cross wires on the rings and in reading the vernier scale.
2. The fine screw may suffer from backlash error. To avoid this error, move the cross wire using screw only in one direction.
3. Error may occur during the measurement of R.

Viva-Voce

1. What are Newton's rings?

   Answer: When a plano-convex surface is placed on a glass plate, an air film of gradually increasing thickness is formed between the two. When monochromatic light is allowed to fall normally on the film and viewed with reflected light, the interference takes place and alternate dark and bright circular fringes are observed. These circular fringes called Newton's rings.

2. State the necessity of large radius of curvature of lens.

   Answer: A large radius of curvature of lens is needed, so that the angle of the air film becomes small. As a result the rings observed have then a comparatively large diameter. Also, it (large radius of curvature of the plano-convex) makes α approximately equal to zero as otherwise Newton's rings will not be observed.

3. What are circular rings?

   Answer: These rings are loci of constant thickness of the air film, these loci being concentric circle; hence fringes are circular.

4. What is the need for an extended source?

   Answer: To view the whole air film, an extended source is necessary.

5. Why do rings get closer as the order of the rings increases?

   Answer: This is due to the fact that the radii of dark rings are proportional to square root of natural numbers while those of bright rings are proportional to square root of odd natural numbers. (*The observations near point of contact should not be taken because the settings of the cross wire in the middle of these bright rings is somewhat difficult due to their large width.*)

6. Why does width of rings decreases with order?

   Answer: In case of wedge-shaped film, the fringe width is a function of $\alpha(\beta = \lambda/2\mu\alpha)$. As the upper surface of film is curved (being the lens surface), α increases with the order of rings, hence fringe width decreases.

7. What do you understand by dark center?

   Answer: At the point of contact, the two interfering rays are opposite in phase (due to additional phase of π) and produce zero intensity.

8. Why is sometimes the center bright?

   Answer: Due to dust particles coming between the two surfaces at the point of contact, as a result effective phase difference between the interfering rays at the point of contact is zero, which is a condition of bright ring.

   Or

   Light rays bend around the dust particle and reach to the central dark ring.

9. What is a microscope?

   Answers: Microscope is an optical instrument that is used to see magnified images of tiny objects. It can produce magnification of the order of 1000. It consists of two convex lenses: Objective (near the object) and Eye lenses (toward the eye) having short focal length and small aperture and large focal length and large aperture, respectively.

10. Explain the different eyepieces.

    Answer: Huygen's, Ramsden's and Gauss (modified form of Ramsden's) eyepieces. Huygen's eyepiece is superior to Ramsden's but not used in lab because the position of cross wires is as if used, it is to be put midway between the two lenses. This involves mechanical difficulty. However, in Ramsden's eyepiece, cross wires are placed outside the eyepiece, hence no mechanical difficulty arises. Therefore, Ramsden's eyepiece is being used in measuring instruments.

Experiment 2

Date: ____________

Objective

To determine the wavelength of a monochromatic light (sodium light) with the help of Fresnel's bi-prism.

Apparatus

Optical bench, bi-prism, micrometer with eyepiece and a monochromatic source (sodium lamp).

Theory and Formula Used

A bi-prism may be regarded as a make-up of two prisms of very small refracting angles placed base to base. In actual practice, a single glass plate is suitably grinded and polished to give a single prism of obtuse angle 179° leaving remaining two acute angles of $30'$ each. This bi-prism device is used to produce two virtual coherent sources for interference from one actual source by dividing wave fronts.

The optical bench used in the experiment consists of a heavy cast iron base supported on four leveling screws. There is a graduated scale along its one arm. The bench is provided with four uprights, which can be clamped anywhere on bench and the position can be read by means of vernier attached to it. Each of the uprights is subjected to the following motions:

1. Motion along bench,
2. Transverse motion (motion at right angle to bench),
3. Rotation around the axis of the upright,
4. With the help of a tangent screw, the slit and bi-prism can be rotated in their own vertical plane.

The arrangement of apparatus is shown in Fig. 1.

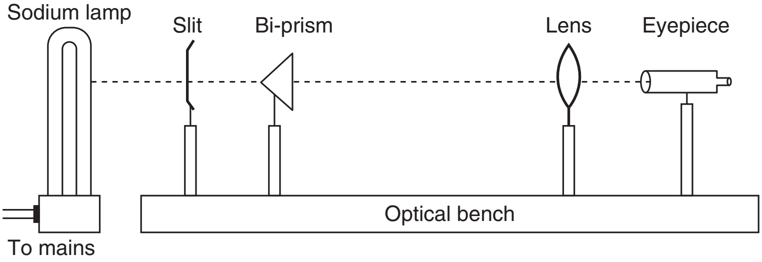


Figure 1

Monochromatic light from a source S falls on two surfaces of the bi-prism, which on refraction bends toward the base. These refracted lights appear to come from S_1 and S_2 (Fig. 2) and interfere and give interference pattern. The interference fringes are hyperbolic, but due to high eccentricity, they appear to be straight lines in the focal plane of eyepiece. The wavelength of sodium light is expressed by

$$\lambda = \beta \frac{2d}{D} \text{Å}$$

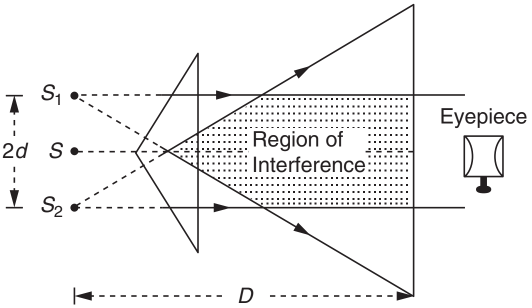


Figure 2

where β is the fringe width, D the distance between slit and screen (in this case eyepiece) and $2d$ is the separation of two virtual coherent sources and can be obtained by deviation method using following formula:

$$2d = 2x\ (\mu - 1)\alpha$$

where x is distance between slit and bi-prism, α is acute angle of bi-prism and μ is the refractive index of bi-prism.

Method

Make one of the cross wires exactly vertical by rotating eyepiece around its own axis. With the help of the tangent screw, make the slit vertical by rotating it in its own plane. The slit is illuminated with the monochromatic source of light, that is, sodium lamp (in this case), and made narrow. Similarly, with the help of tangent screw provided at bi-prism stand, make the edge of the bi-prism parallel to the slit. By moving the bi-prism stand laterally with the help of the screw provided at the side of the stand and observing through bi-prism, make two virtual coherent sources clearly visible. Now by adjusting the slit stand, bi-prism stand, and eyepiece stand, with the help of stand side screws providing for lateral motion, make all the stands in a straight line. Now in this situation the clear narrow fringes are observed through eyepiece. The fringes can be seen more distinct and clear by moving eyepiece away from the slit (i.e., by increasing D).

If the line joining the slit and the central edge of the bi-prism are not parallel to the length of the bench, fringes would shift laterally as the eyepiece moves away from slit. To overcome this problem, move the bi-prism a small distance transversely to the bench, in a direction opposite to the direction of the shifting of fringes until this lateral shift vanishes.

For the measuring fringe width, set the cross wire at the center of the any bright fringe and the reading of the micrometer screw is taken. Now move the cross wire in one direction so that the cross wire falls in succession at the canters of the bright fringes (say 11th fringe) and take the corresponding readings. Hence, calculate the widths of a number of fringes (say 10) by subtracting the first fringe from 11th and so on. After taking the mean, fringe width β for one fringe is calculated.

The distance between the eyepiece and the slit is measured as D and between slit and bi-prism as x. Knowing the value of α and μ, the wavelength of sodium light can be calculated.

Observation

For the fringe width (β)

$$\text{Least count of the micrometer screw} = \underline{\qquad\quad}\ \text{cm}\left(\text{LC} = \frac{\text{Pitch}}{\text{Total no. of divisions on circular scale}}\right)$$

$$\text{Pitch} = \frac{\text{Distance move along the linear scale}}{\text{No. of complete rotations given to the circular scale}}$$

Sl. No.	*First Micrometer Reading for Any Fringe* (x)	*Second Micrometer Reading for n Fringes* (y)	*Separation of n Fringes* ($y - x$)	*Fringe Width* (*cm*) $\beta = (y - x)/n$
1				
2				
3				Mean fringe width (cm) β = _______

Positions of upright carrying slit (a) = _______________ cm
Positions of upright carrying bi-prism (b) = _______________ cm
Positions of upright carrying the eyepiece (c) = _______________ cm
Value of D [(c) – (a)] = _______________ cm
Value of x [(b) – (a)] = _______________ cm

Calculations

The separation between coherent sources is $2d = 2x(\mu - 1)\alpha$

The wavelength of monochromatic light is $\lambda = \beta \dfrac{2d}{D}$

Conclusion

Wavelength of sodium light is found to be λ = _________ Å

$$\%\ \text{error} = \frac{(\text{Actual value} - \text{Calculated value}) \times 100}{\text{Actual value}}$$

Possible Errors

1. Error can take place in adjusting vertical and narrow slit.
2. There may be an error in making bi-prism edge parallel to the slit.
3. Measuring fringe width at a close distance may also create error.

Viva-Voce

1. What is interference?

   Answer: When two waves superimpose, resultant intensity is modified. The modification in the distribution of intensity in the region of superposition is called interference. There is no loss of energy in interference, but only there is redistribution of energy, that is, energy from dark places is shifted to bright places.

2. What are interference fringes?

   Answer: There are alternate straight bright and dark patches of light obtained in the region of superposition of two wave trains of light.

3. What happens with small refracting angles of the bi-prisms?

   Answer: Due to small refracting angles of the bi-prisms, fringe width will be large.

4. What is the purpose of the bi-prism?

   Answer: The purpose of the bi-prism is to produce two coherent virtual sources from a given source (slit). Two coherent sources are situated symmetrically on either side of the slit and in the same plane as the slit.

5. What is the nature of fringes?

   Answer: The fringes are hyperbolic but due to large eccentricity of the hyperbola, they appear as straight fringes.

6. How should the cross wire be adjusted?

   Answer: Basically, cross wire should be adjusted in the middle of the fringe; however, it is difficult to adjust the cross wire in the middle of each fringe accurately, so it is suggested to coincide the cross wire with the left or right edge of each fringe.

7. What is lateral shift?

   Answer: The shifting of the fringes across the optical bench when eyepiece is moved toward or away from the bi-prism, is called the lateral shift. It shows that the line joining the center of the slit and the edge of the bi-prism is not parallel to the bed of the optical bench.

8. What is the effect of replacing monochromatic light by white light source?

   Answer: In case of white light, the interference pattern consists of a central white fringe in between a few colored fringes.

9. What is zero-order fringe?

   Answer: First of all the interference fringes are observed with monochromatic light. Then monochromatic light is replaced by white light. Now the cross wire is fixed at center white fringe. Again, replace the white light source by monochromatic light. The fringe on cross wire is a zero-order fringe.

10. What is monochromatic source?

    Answer: It is a source of light that produces single wavelength. Although sodium light is not perfectly monochromatic (it has two close wavelengths 5890 Å and 5896 Å), but due to small difference of 6 Å it is practically treated as monochromatic light with wavelength 5893 Å. Laser light is perfectly monochromatic source of light.

11. What is the effect of blue light?

    Answer: Due to small wavelength of blue light fringe width decreases as a result fringes become denser.

12. What is the effect of slit width?

    Answer: As the fringes are formed in the entire region between the bi-prism and eyepiece, so if we increase the slit width the fringes become poorer and poorer.

Experiment 20

Date: ____________

Objective

To draw hysteresis curve (B–H) curve of a specimen in the form of a transformer and to determine its hysteresis loss.

Apparatus

Cathode–Ray oscilloscope (CRO), potentiometeric resistor, specimen in the form of transformer, rheostat, condenser, AC voltmeter (0–10 V) AC milliammeter.

Theory and Formula Used

The behavior of ferromagnetic materials in which magnetization lags behind the magnetizing field is known as hysteresis. In other words, we can say that when ferromagnetic materials are subjected to external magnetic fields then materials are magnetized in one direction and then in reverse direction. If we plot a graph between intensity of magnetization and magnetizing field (H), it may be represented by a curve known as hysteresis curve and the area of the closed curve is equivalent to hysteresis loss.

When a current I (ampere) flows through the primary of a transformer, the hysteresis loss per unit volume per cycle is given by

$$W = \frac{iV \cdot \text{area of } B\text{–}H \text{ curve}}{f\pi \cdot \text{area of the rectangle}} \text{ J/cycle}$$

Method

First, make the connection as shown in Fig. 1. Switch on the CRO and adjust it properly for an intense point on the screen. The voltage V from the output of the transformer is applied to the y plate and the frequency selector of CRO is switched on to external position. The vertical and horizontal gains are so selected that B–H curve is obtained on the screen as shown in Fig. 2. After that, plot a B–H curve on a tracing paper. Similarly, we can also obtain different B–H curves for different set of values of V and i. Through graph paper, we can calculate the area of B–H curve and its rectangle in mm^2.

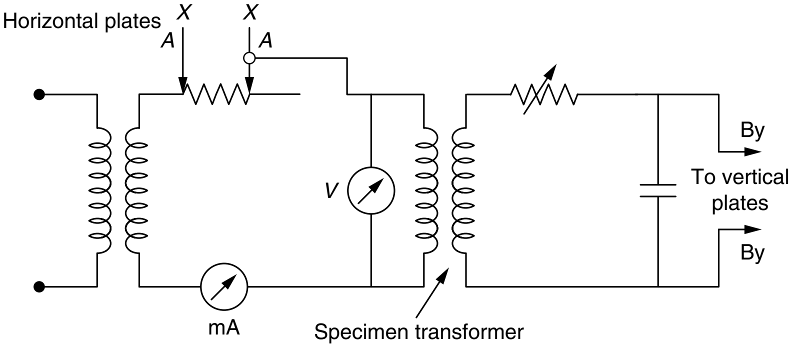


Figure 1

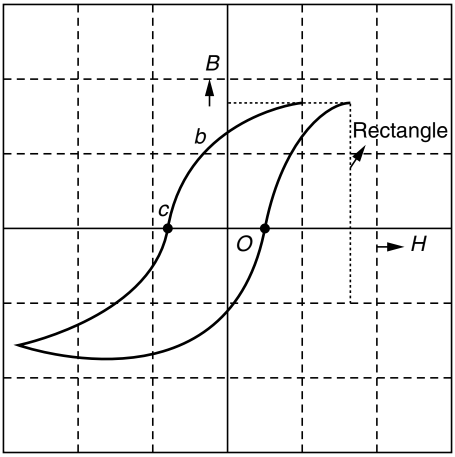


Figure 2

Observations

Sl. No.	Current I (mA)	Voltage V (V)	Area of B–H Curve (mm^2)	Area of the Rectangle (mm^2)
1.				
2.				
3.				
4.				
5.				
6.				
7.				
8.				

Calculation

Hysteresis loss per unit volume per cycle is calculated from

$$W = \frac{iV \cdot \text{area of } B\text{–}H \text{ curve}}{f\pi \cdot \text{area of the rectangle}} = \underline{\qquad\qquad} \text{ J/cycle}$$

Conclusion

The hysteresis loss of the specimen per unit volume per cycle is ________ J/cycle

Possible Errors

1. The applied voltage to the CRO plates must be stabilized.
2. Horizontal and vertical gain knobs should be unchanged throughout the experiment.
3. The intensity of the spot should be very intense on the CRO screen.

Viva-Voce

1. What is hysteresis?

   Answer: The behavior of ferromagnetic materials in which magnetization lags behind the magnetizing field is known as hysteresis.

2. What is hysteresis loss?

   Answer: In absence of an external magnetic field, ferromagnetic materials are spontaneously magnetized but in random direction. Therefore, net magnetization is zero. However, in the presence of magnetic field, domains align along the direction of electric field and absorbs energy. When field is removed, the domains do not get back and hence energy is not recovered. Thus, retained energy by the specimen, dissipated in terms of heat, is known as hysteresis loss.

3. What is retentivity?

   Answer: The magnetism that retains the material after removing the magnetizing field is called retentivity or remanence.

4. What is coercivity?

   Answer: If the direction of the magnetizing field is increased in negative direction of the field then magnetization traces the path bc and becomes zero at point *c*; the value *oc* is called coercivity or coercive force of the material (see Fig 2).

5. What is the use of hysteresis curve?

   Answer: Hysteresis curve is used for the selection of materials to make permanent magnet, electromagnet or transformer cores.

6. What is magnetization (I)?

   Answer: The total magnetic moment per unit volume of the material is known as magnetization or intensity of magnetization. Mathematically

$$I = \frac{\mu_m}{V}$$

   and its unit is Ampere-turn/meter.

7. What is magnetic susceptibility?

   Answer: It is the ratio of magnetization I to the magnetic field H and denoted by $\chi_{m.}$

$$\chi_m = \frac{\vec{I}}{\vec{H}} \qquad \text{For vacuum } \chi_m = 0$$

Experiment 3

Date: ____________

Objective

To determine the wavelength of different spectral lines of mercury light using plane transmission grating.

Apparatus

Spectrometer, diffraction grating, source of spectrum (mercury light), prism and a reading lamp.

Theory and Formula Used

A diffraction grating is an arrangement of N similar parallel slits each of width "e" and separated by equal opaque spaces "d". A plane wave front of wavelength "λ" light is incident normally on the grating. According to the Huygen's theory, each point in each slit sends secondary wavelets in all possible directions (see Fig. 1). Let us find the resultant of wavelets diffracted at an angle θ with the normal. Secondary disturbances issuing from each slit can be replaced by central disturbances C_1D_1, C_2D_2, …, C_nD_n, each of constant amplitude $R_o = A\,(\sin\alpha/\alpha)$ in the increasing phase in arithmetic progression by an amount of

$$\delta = \frac{2\pi}{\lambda} C_2K_2 = \frac{2\pi}{\lambda}(e+d)\sin\theta = 2\beta \text{ (suppose)} \tag{1}$$

The resultant OP_N of these waves can be found by the vector diagram of regular polygon (Fig. 2) whose sides OP_1, OP_2, … are of equal lengths ($=R_o$) and suffers a successive inclinations δ. If C is the center of polygon, then

$$OP_1 = 2\times OC\times\sin(\delta/2) = R_o \qquad [\text{since } \angle OCP_1 = \delta]$$

$$OP_N = 2\times OC\times\sin(N\delta/2) = R \qquad [\text{since } \angle OPC_N = N\delta]$$

The total resultant R is then

$$R = R_o\left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right] = A\frac{\sin(\alpha)}{\alpha}\cdot\frac{\sin(N\beta)}{\sin(\beta)}$$

And the resultant intensity I that is proportional to R^2 is

$$I = I_o\left(\frac{\sin\alpha}{\alpha}\right)^2\cdot\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \qquad [R_o{}^2 = I_o] \tag{2}$$

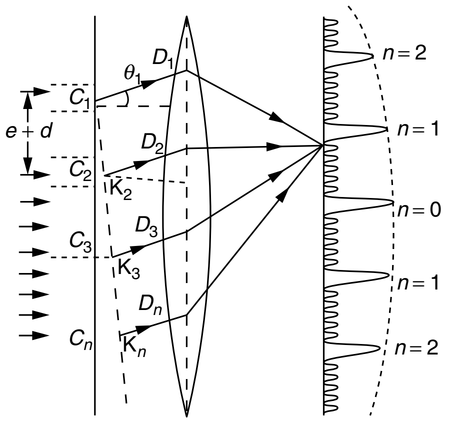


Figure 1

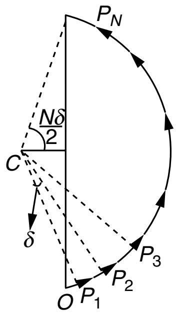


Figure 2

The first factor of Eq. (2) gives the intensity distribution due to diffraction at the individual slit, while the second one gives the interference pattern between N diffracted waves from N slits. Principle maxima occur when the interference term becomes maximum if

$$\sin\beta = 0 \quad \text{or} \quad \beta = \pm n\pi \tag{3}$$

where $n = 0, 1, 2, 3$

$$(e+d)\sin\theta = \pm n\lambda \tag{4}$$

The same condition holds for the reinforcement of the vibrations from C_3 and C_4, etc. Thus in the direction θ, a bright image of the slit is obtained. There are several maxima called principle maxima that are satisfying the above condition (4).

If grating element $(e+d)$ and n are constant, sin θ and therefore θ varies directly as λ. If the incident light emits several fixed wavelengths (in mercury light), after diffraction, the maxima of different wavelengths are seen in different directions and thus the incident light is split-up into its spectrum. At $\theta = 0$, all the rays are in the same phase of vibration, so the central image has the same color as the source. For other value of θ satisfying Eq. (1), there are spectra of first order (from $n = \pm 1$), second order (from $n = \pm 2$), etc. of falling intensities on either side of the central maximum.

If N be the total numbers of lines per inch on the grating, then

$$(e+d) = 2.54/N \text{ cm}$$

Thus, the expression (4) can be written as

$$N\lambda = \frac{2.54}{N}\sin\theta$$

Minima occurs when $\sin N\beta = 0$ (but $\sin\beta \neq 0$) or $N\beta = \pm m\pi$ where m has any integral value but not 0, N, $2N \ldots nN$. Otherwise, it will correspond to principal maxima given by (3). Hence for principal maxima

$$N(e+d)\sin\theta = (0, N, 2N, \ldots, nN) \tag{5}$$

[which may also be obtained by multiplying both sides of Eq. (4) by N] and for minima

$$N(e+d)\sin\theta = \lambda, 2\lambda, \ldots, (N-1)\lambda, (N+1)\lambda, \ldots, (2N-1)\lambda, (2N+1)\lambda, \ldots, (nN-1)\lambda, (nN+1)\lambda \tag{6}$$

Method

Adjust the level of turntable and focus the telescope and collimator using Schuster's method. Now see the slit through telescope and collimator without using diffraction grating and focus it on cross wire of eyepiece. Rotate the telescope 90° toward left direction and place the diffraction grating on the turntable (see Fig. 3).

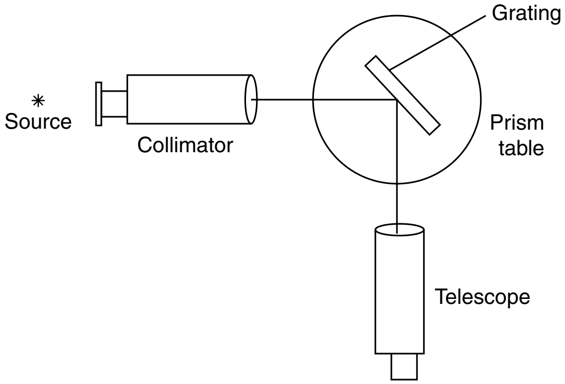


Figure 3

Adjust the grating by rotating the turntable without touching the telescope such that the slit again appears at the cross wire of eyepiece through reflection from diffraction grating. When slit is seen clearly, rotate the turntable by 45° toward right so that the diffraction grating becomes normal to the incident light and the ruled surface faces the telescope. Bring the telescope again on its original positions by rotating it 90° toward right. This is the position of normal incidence where slit appears at cross wire through grating.

Now rotate the telescope slowly in either side (say left) from normal position and focus it at particular color. It is the first color of first-order spectrum. Take the measurement at this position on both the vernier scales (say v_1 and v_2), turn the telescope to the other side (say right) and focus it for the same color and take readings again on both scales. The difference of the same vernier scale (say v_1) will give you double of the diffraction angle for that particular color. The similar measurement can be obtained for the other colors as well. The second-order spectrum can also be viewed by rotating the telescope further beyond the first-order spectrum toward left or right and so for other higher orders of spectrum as well provided grating is able to produce it.

Observations

Least count of the spectrometer = _______ s.

Color of Line	*Reading for Spectrum on Left Side*		*Reading for Spectrum on Right Side*		*Average Difference of the Reading of the Same Vernier (2θ)*		*Mean (2θ)*	*Mean (θ)*
	Ver. A	*Ver. B*	*Ver. A*	*Ver. B*	*Diff. Ver. A*	*Diff. Ver. B*		
Violet								
Green								
Yellow								

Calculation

The wavelength of different spectra lines is given by

$$\lambda = \frac{2.54}{N} \sin\theta$$

where N is total number of lines per inch on diffraction grating.

For ______ color, $\lambda =$ ______ Å.

Conclusion

The wavelength of the violet, green and yellow colors is, respectively, ______ Å, ______ Å and ______ Å.

Possible Errors

1. Error may arise in setting the normal incidence.
2. There may be an error in setting the cross wires on the image.
3. It may take place in reading the vernier scale.

Viva-Voce

1. What is diffraction of light?

   Answer: When light falls on an obstacle or small aperture whose size is comparable with the wavelength of light, there is a departure from straight line propagation; the light bends round the corners of obstacle or aperture. This bending of light is called diffraction.
2. What is the difference between interference and diffraction?

   Answer: Interference of light takes place due to the superposition of two ways coming from two different coherent sources, while diffraction is due to the mutual interference of secondary wavelets originating from the various points of the wavefront that are not blocked off by the obstacle.
3. What is diffraction grating?

   Answer: An arrangement consisting of a large number of parallel slits of same width and separated by equal opaque spaces is known as diffraction grating. The distance between the centers of two successive slits is called grating element. This is denoted by $(a+b)$, where a is the width of transparent part and b is the width of opaque part.
4. What is the difference between prism spectrum and a grating spectrum?

   Answer: In grating spectrum red color is deviated most and violet the least, while this order is reversed in prism spectrum. Prism spectrum is more intense than the grating spectrum because the light is concentrated in one spectrum while in case of grating, the incident light is diffracted into spectra of various orders; moreover most of the light is concentrated in direct image where no spectrum is formed.
5. What is dispersive power of grating?

   Answer: The rate of change of angle of diffraction with wavelength is defined as the dispersive power of grating. Dispersive power is more for higher orders. It depends on only material of the prism and is independent of refracting angle of prism.
6. Which order of spectra is possible?

   Answer: Only first- and second-order spectra are possible. Third-order spectrum is not possible because the maximum value of $\sin\theta = 1$, and according to Eq. (4) $n_{\max} = 2.874$, which is less than 3.

Experiment 4

Date: ____________

Objective

To determine the specific rotation of cane sugar solution using biquartz polarimeter.

Apparatus

Biquartz polarimeter, ordinary white light source, measuring flask, beaker, cane sugar and distilled water.

Theory and Formula Used

The plane polarize light on passing through optically active substances/solutions rotates by certain angle. The rotation depends on the types of substance and the wavelength of light used. When this rotation produces one-decimeter length of solution of unit concentration, we call it specific rotation. The specific rotation does not depend on length of solution or concentration of solution, but on temperature and wavelength.

As shown in Fig. 1, *S* is the source of ordinary white light placed at the focus of convex lens, so that beam becomes parallel after passing through lens. Light then passes through the Nicol Prism that acts as polarizer. The polarized light passes through the biquartz and travels over the length of the polarimeter tube made of glass. The light is analyzed with the help of the analyzer, the second Nicol Prism, which can be rotated about a horizontal axis. Its position can be read by a vernier moving over a fixed circular graduated scale. The light now is viewed with the help of a telescope. The analyzer and telescope are placed in the same tube.

A biquartz plate consists of two semi-circular plates of right handed and left handed quartz. The thickness of each plate is chosen to be near about 3.75 mm so that the yellow light ($\lambda = 5900$ Å) be rotated through 90°. These plates are cut in such a way that the optic axis lies at right angles to its faces. When the plane polarized light is incident normally on the biquartz plate, along N_1AN_1 (principal section of polarizer), component colors are rotated through different angles where it is maximum for violet, least for red and 90° for yellow (Fig. 2). In the two halves, the colors rotate symmetrically in opposite direction. The planes of vibration of the different colors, that is, red, orange, yellow, blue and violet, are represented,

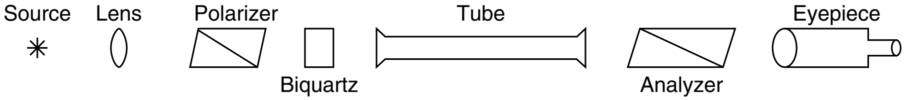


Figure 1

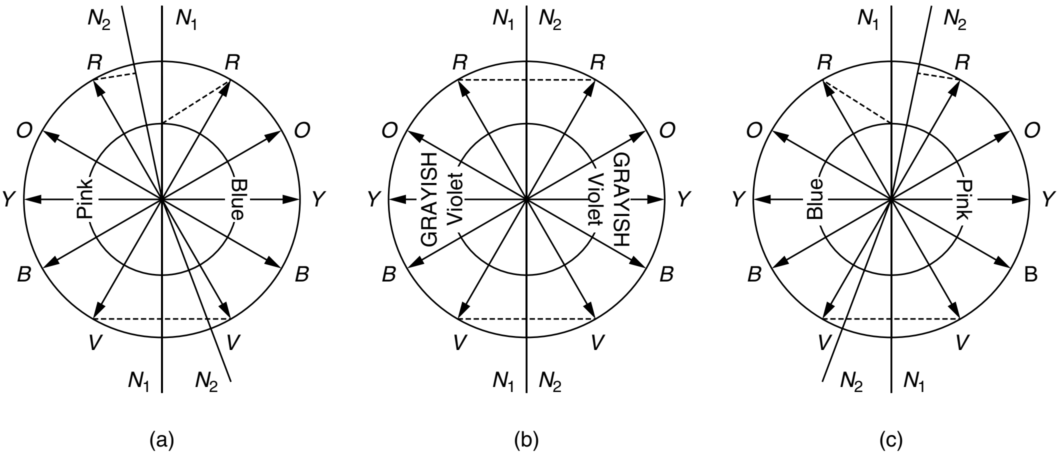


Figure 2

respectively, by *AR*, *AO*, *AY*, *AB* and *AV*. When the principal section of the analyzer N_2AN_2 is parallel to N_1AN_1 [Fig. 2(b)], the wavelength of the yellow color is completely quenched from both the halves. As all the other colors would be inclined equally in both the halves, the resultant colors produced in each half would be the same. This resultant is greenish-violet called "sensitive tint" [Fig. 2(b)]. If the analyzer slightly rotates from this setting in the anti-clockwise direction, the transmitted components of the longer wavelengths (predominantly red) increase while those of the shorter wavelengths (predominantly violet and blue) decrease. The left half will appear pink and right half will appear blue. The appearance is reversed when the analyzer is rotated in the clockwise direction [Figs. 2(a) and (c)].

The specific rotation of the plane of polarization of solution (sugar dissolved in water) can be determined by the following formula:

$$S = \frac{\theta V}{lm} \ldots \text{degree/conc./dm}$$

where θ is the rotation produced in degrees, l is the length of the tube in decimeter, m is the mass of sugar in g dissolved in water and v is the volume of sugar solution.

Method

Fill the polarimeter tube with water minimizing the air gap and place it in proper position. The slit is illuminated with white light (with sodium light in case of half shade polarimeter) and focus the eyepiece such that the field of view is sharp. The unequal illuminated columns may be seen in the two halves of the field of view. Rotate the analyzer until the two half portions or colors change into one homogeneous color. Take the measurement on circular and vernier scales. Rotate the analyzer about 180° for same color and take the measurements again.

Prepare the sugar solution of known concentration (say 10 g in 100 ml of distilled water). Fill the polarimeter tube with sugar solution, repeat the above procedure and note down the measurement in similar way. The difference in the observations taken in distilled water and in sugar solution is the angle of rotation (θ) for that concentration. Repeat the experiment by changing concentration of sugar solution and take at least five sets of readings. Plot the graph between concentration and angle of rotation. It will be a straight line.

Observations

Room temperature = ______ °C
Length of the tube (l) = ______ dm
Mass of the sugar dissolved (m) = ______ g
Volume of the solution = ______ cm^3
Least count of the vernier = ______

Sl. No.	***Reading on Analyzer for Equal Illumination in Degree***					***Rotation in Degree***		
	With Distilled water		***With Sugar Solution***			θ_1	θ_2	$\frac{\theta_1 + \theta_2}{2}$
	First Position A	***Second Position B***	***Strength of the Solution***	***First Position C***	***Second Position D***	(A–C)	(B–D)	
1								
2								
3								
4								
5								

Calculation

Draw a graph between concentrations and angle of rotation (θ). It will be a straight line curve as shown in Fig. 3. From the graph, find out the value of θ for a particular concentration. Then using following formula calculate specific rotation.

$$S = \frac{\theta V}{lm}$$

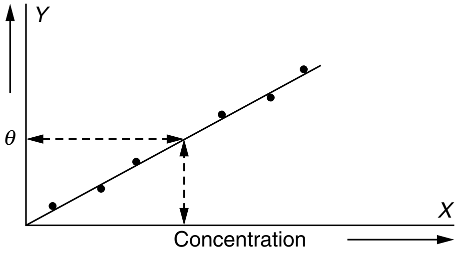


Figure 3

Conclusion

The graph between concentration and angle of rotation suggests that the angle of rotation of plane polarized light is proportional to the concentration of solution. The specific rotation of cane sugar solution at °C is = __________ degree per unit concentration per decimeter.

Possible Errors

1. Error is possible in measuring the mass of sugar for the preparation of sugar solution.
2. Error may occur in taking the reading of vernier scale.
3. The two colors at 180° may not be exactly same as viewed by eyepiece.

Viva-Voce

1. What is polarimeter?

   Answer: It is an instrument (polarizer, analyzer and glass tube having large diameter at middle ensures that there is no bubble in the path of optical ray) that is used to measure the angle of rotation through which the plane of polarization is rotated by an optically active substance. Laurent's half shade and biquartz are the types of polarimeter.
2. What is polarized light?

   Answer: The light, which has acquired the property of one-sidedness, is called a polarized light.
3. How does polarized light differ from ordinary light?

   Answer: The ordinary light is symmetrical about the direction of propagation while in case of polarized light, there is lack of symmetry about the direction of propagation.
4. What is the difference between plane of vibration and plane of polarization?

   Answer: The plane containing the direction of vibration as well as the direction of propagation of light is called the plane of vibration. On the other hand, the plane passing through the direction of propagation and containing no vibration is called plane of polarization.
5. What is double refraction?

   Answer: When ordinary light is incident on a calcite or quartz crystal, it splits into two refracted rays (O-ray and E-ray with vibrations in two mutually perpendicular planes) and this phenomenon is known as double refraction.
6. What is Nicol prism?

   Answer: It is an optical device invented by William Nicol, to produce and analyze the plane polarized light. Its principle is based on the phenomenon of double refraction. Nicol prism eliminates ordinary-ray by the phenomenon of total internal reflection and allows transmitting extra ordinary-ray.
7. What is specific rotation?

   Answer: For a given wavelength of light and at a certain temperature, specific rotation is defined as the rotation produced by one dm length of the solution containing 1 g of optically active substance per cubic centimeter of solution. It is represented in degree $(\mathrm{dm})^{-1}(\mathrm{g/cm^3})^{-1}$.

Experiment 5

Date: ____________

Objective

To determine the focal length of the combination of two lenses separated by a distance, and to verify the formula for the focal length of combination of lenses.

Apparatus

Nodal slide assembly bench, source of light, metallic screen cross slit, a plane mirror and two convex lenses.

Theory and Formula Used

The purpose of this experiment is to determine the focal length of the combination of lenses. In the experiment setup the nodal slide assembly consists of an optical bench provided with four uprights as shown in Fig. 1. The one upright carries a bulb placed in a metallic cover having a circular aperture, which illuminates a cross slit in the adjacent upright. The third upright carries a nodal slide. Nodal slide is essentially a horizontal metal support capable of rotation around a vertical axis, and lens or lenses can be mounted upon the support. The metallic support can be fixed or it can be moved back and forth by means of a screw so that the relative positions of the two lenses can vary with respect to this upright. The support can be rotated in a horizontal plane. The fourth upright carries a plane mirror that can be rotated around a horizontal axis perpendicular to the bed of the bench.

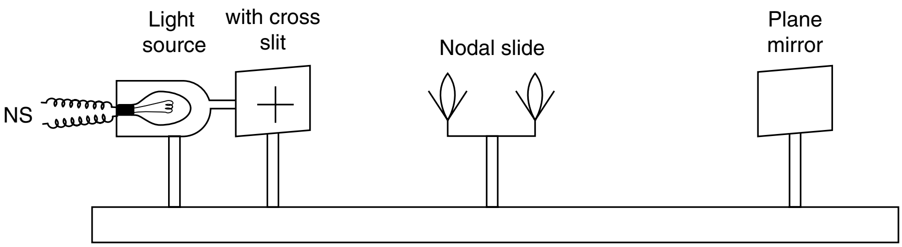


Figure 1

If parallel beam of light is incident on a converging lens system thus forming an image on screen in its second focal plane, the image does not shift laterally when the system rotated around a vertical axis passing through its second nodal points. The principle is based on the property of nodal points, that is, when a ray of light passes through the other and is always parallel to the incident ray. If the system is now rotated slightly around a vertical axis, the image will not be shifted from its position as shown in Fig. 2. The distance of the screen from the axis of rotation gives the principal focal length of the lens system.

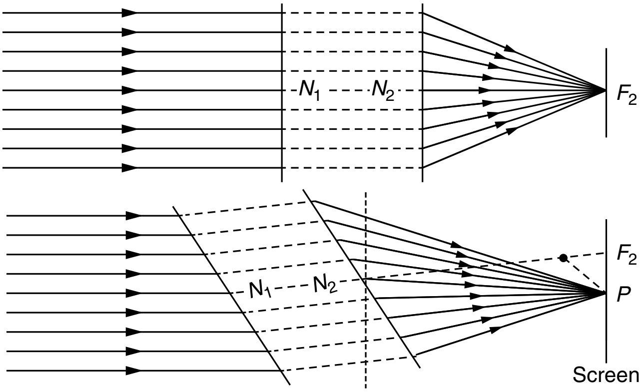


Figure 2

The focal length of the combination of lenses is

$$\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2} \tag{1}$$

where f_1 is the focal length of first single lens, f_2 the focal length of another lens, F is the focal length of combination of lenses and d is the distance between two lenses.

Method

Adjust the lamp, cross slit, lenses and mirror in such a way that their axis lies along the same horizontal line. Lenses should be kept at a known separation (*d*). The nodal slide is adjusted so that both lenses are at equidistance from the center of stand so that the distance of lenses can be easily measured on bench scale. Now the lens system as a whole is moved toward or away from the slit until a clear and proper image of the cross slit is formed on the same screen of slit. In this position the distance between slit and center point of combination lens is the focal length (*F*) of the combination of lenses. The exact position can be judged by rotating the nodal carriage about 5° from its position, the image of cross slit should not move. Now rotate the nodal carriage by 180° and repeat the above procedure. The procedure is also repeated above by changing the distance between lenses. In a similar way one can also find the focal length of single lens (i.e., f_1 and f_2).

Observations

Table for focal length of a lens:

Sl. No.	*Light Incident on*	*Distance Between Lenses (d) cm*	*Position of Cross Slit (a) cm*	*Position of Nodal Stand (b) cm*	*Focal Length of the Lens (a− b) cm*	*Average Focal Length cm*
First	One face					
	Other face					
Second	One face					
	Other face					
Combined	One face					
	Other face					
	One face					
	Other face					
	One face					
	Other face					

Calculation

Equation (1) can be used to estimate the focal length F.

Conclusion

To compare calculated and observed focal lengths we tabulate as follows:

Sl. No.	*Separation (cm)*	*Observed (F) cm*	*Calculated (F) cm*
1			
2			
3			

The above table clearly shows that the calculated value of the focal length of combination is approximately same as the experimental. Hence, the formula $\frac{1}{F}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{d}{f_1 f_2}$ is verified.

Possible Errors

1. Error is possible in focusing the image of cross slit.
2. Error may occur in reading the positions of the mark on the lens holder.

Viva-Voce

1. What is nodal slide?

   Answer: It is a device to determine the nodal points of a lens system. It holds the lenses and can be rotated about a vertical axis.
2. What are nodal points?

   Answer: Nodal points are a pair of conjugate points having unit positive angular magnification.
3. What are cardinal points?

   Answer: Three pairs of an optical system called focal points (a pair of points lying on the principal axis and conjugate to points at infinity), principal points (a pair of conjugate points on the principal axis of the coaxial optical system having positive linear magnification) and nodal points of a coaxial system are called cardinal points.
4. What is principle?

   Answer: If a parallel beam of light is incident on a converging lens system thus forming an image on a screen in its second focal plane, the image does not shift laterally when the system rotated around a vertical axis passing through its second nodal points. Thus, the principle is based on the property of nodal points.
5. What is desired image?

   Answer: If by rotation of plane mirror the image moves on the screen then it is the desired image otherwise not. The position of the desired image will not be affected in any way by alternating the position of plane mirror.
6. What is the rotation of lens combination by 180°?

   Answer: Need to rotate the combination, because the two nodal points are not symmetrical with respect to two lenses. It is not necessary if the focal length is same for the two lenses.
7. What is the importance of cardinal points?

   Answer: If the cardinal points of a coaxial system of lenses are known, the image formation becomes very easy. In this case, the refraction at each surface of the lens is not considered.
8. What is the reduction of cardinal points?

   Answer: Since, the medium on either side of the optical system and between the lenses is same (or air), the nodal points coincide with the principal points. Therefore, the six cardinal points reduce to four.
9. What is the function of plane mirror?

   Answer: Plane mirror reflects the parallel rays of light in the same path falling on it.

Experiment 6

Date: ____________

Objective

To determine resistance per unit length and specific resistance of the given resistance using Carey Foster's bridge.

Apparatus

Carey Foster's bridge, Leclanché cell, resistance box, rheostat, potentiometer, jockey, connecting wire, the unknown resistance and a galvanometer.

Theory and Formula Used

Carey Foster's bridge is a modified arrangement of Wheatstone's bridge. This bridge is more sensitive due to two reasons: (a) The effective length of the wire is increased without actually using a wire of more than 1 m length and (b) effects due to end corrections are eliminated.

The resistance per unit length (K) of the bridge wire is given by the formula

$$X - Y = K(l_2 - l_1) \tag{1}$$

where X (known) and Y (unknown) are two resistances in the outer gaps of the Carey Foster's bridge and l_1 and l_2 denote the length of the balance point on the bridge wire before and after interchanging the resistances X and Y. If $Y = 0$ (i.e., copper strip) and $X = R$ (known resistance), then we have

$$K = \frac{R}{l_2 - l_1} \tag{2}$$

Using Eq. (2), the resistance per unit length (K) can be calculated and knowing the value of K, the unknown resistance Y can be calculated.

Method

The connections are made as shown in Fig. 1 such that potentiometer acts as two equal resistance P and Q. Now connect the copper strip in the right gap of the bridge (as resistance Y) and a known resistance X (=R) in the left gap. The null point is determined and its distance l_1' from the left end of the bridge is measured. Interchange the positions of R and copper strip and note down the distance l_2' of the new null point from

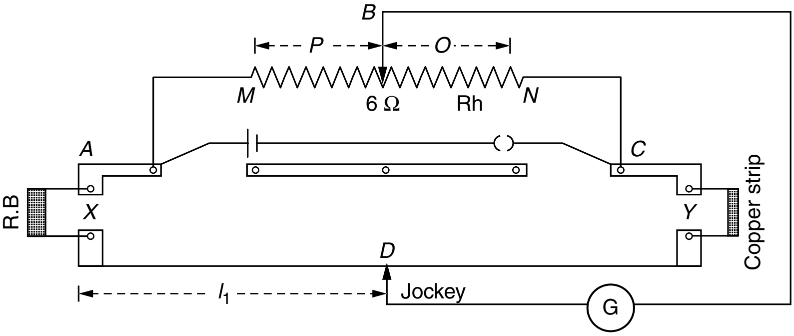


Figure 1

the same left end. Take at least three readings for different values of known resistance *R*. Now the given wire, whose resistance is to be determined (say *Y*), is placed in the left gap in place of copper strip and known resistance in right gap (*X*) find the null point and measure the distance l_1 and similarly l_2 when *Y* and *X* are interchanged. This part should also be repeated for different values of *X*.

Observations

For determination of *K* (resistance per unit length of the bridge wire)

Sl. No.	*R (in Ω)*	*Distance of the Null Point l*		$l_2' - l_1'$ *(in cm)*	$K = \dfrac{R}{l_2' - l_1'}$
		When R in Left Gap l_1' *(in cm)*	*When R in Right Gap* l_2' *(in cm)*		
1					
2					
3					

Average $K =$ ________ Ω/cm

For determination of *Y*

Sl. No.	*X (in Ω)*	*Distance of Null Point*		$l_2 - l_1$ *(in cm)*	*Y*
		When Y in Left Gap l_1 *(in cm)*	*When Y in Right Gap* l_2 *(in cm)*		
1					
2					
3					

Average $Y =$ ________ Ω

Calculation

Calculate K using Eq. (2) and thereafter Y through Eq. (1).

Conclusion

The resistance of unknown wire is = __________ Ω

Possible Errors

1. All connections must be tight to avoid any additional resistance.
2. The cell circuit should be closed only when readings are being taken.
3. In checking the null point, the cell circuit must be completed before the galvanometer circuit.

Viva-Voce

1. What is resistance?

   Answer: The ratio of the potential difference between the two ends of a conductor to the current flowing in it is called the resistance of the conductor. Resistance of a conductor is directly proportional to its length (l), inversely proportional to the area of cross section (A). It also depends upon the nature of material and temperature of the conductor. The resistance of bridge wire increases with the increase in temperature. The minimum difference in resistance that we can measure is equal to the resistance offered by one millimeter length of the bridge wire and maximum difference in resistance is equal to the resistance of the total length of the bridge wire.
2. What is specific resistance?

   Answer: Specific resistance of a substance is the resistance of a piece of that substance having unit length and unit area of cross section. Specific resistance is independent of length or radius of the wire and it is different for different materials.
3. What are materials of wire?

   Answer: Due to high resistivity and low temperature coefficient, constantan (Cu 60% and Ni 40%) or manganin (Cu 60%, Mn 12% and Ni 4%) alloys are used for making the resistance wire.
4. What is the principle of Carey Foster bridge?

   Answer: It is based on the principle of Wheatstone bridge. It is the most sensitive bridge because all the four resistances are equal or at least of the same order.
5. What is resistance box?

   Answer: It has a number of resistance coils of various resistances, connected in series.

Experiment 7

Date: ____________

Objective

To determine electrochemical equivalent (ECE) of copper.

Apparatus

Helmholtz galvanometer, copper-voltmeter, copper plate, power supply, stopwatch, four-way key and rheostat.

Theory and Formula Used

The electrochemical equivalent of copper (z) can be defined by the following formula:

$$z = \frac{m}{it}$$

where i is the current in the circuit and equal to $k \tan\theta$, and m is the mass of copper deposited on cathode plate in time t.

In our experiment we have kept $\theta = 45°$ constant, so $i = k$, that is, equal to the reduction factor of the Helmholtz galvanometer. Hence,

$$z = \frac{m}{kt} = \underline{\hspace{3em}}\ \text{g/C}$$

Method

Make the connections as shown in the circuit diagram (Fig. 1). Set the plane of coils of Helmholtz galvanometer in magnetic meridian. Keeping system ON adjust the current in the circuit using variable resistance such that the deflection in compass box is 45°, which should be kept constant throughout the observation. Now make the circuit OFF. Clean the cathode plate and weigh it. Put the copper cathode again in copper voltmeter and make the circuit ON. Let the circuit ON and current flow at least for 20 min in one direction and 20 min in reverse direction. The direction of current can be changed in the Helmholtz galvanometer with the four ways key. After observing 40 min, time (t) interval, take the copper plate out from copper voltmeter, first air dry and then weigh.

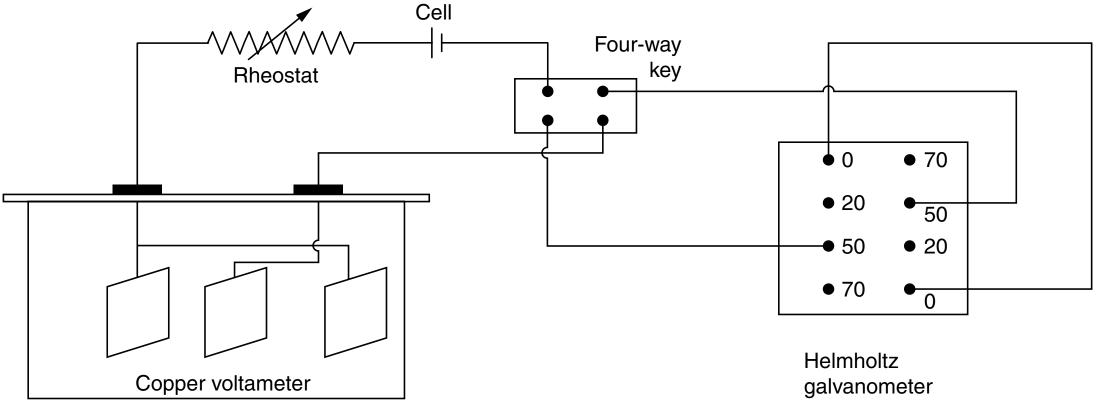


Figure 1

Observations

Mass of the copper plate (a) = ___________ g

Mass of the copper plate + copper deposit (b) = ___________ g

Mass of copper deposited on the copper plate (m) = ___________ g

Copper deposited for the time (t) = ___________ s

Reduction factor of the Helmholtz galvanometer for 50 turns is 0.2

Sl. No.	*Current in Any Direction*		*Current in Opposite Direction.*	
	Time (min)	*Deflection*	*Time (min)*	*Deflection*
1	0	45°	0	45°
2	5		5	
3	10		10	
4	15		15	
5	20		20	

Calculation

The electro-chemical equivalent of copper can be obtained by the following formula

$$z = \frac{m}{kt} = \underline{\qquad\qquad} \text{ g/C}$$

Conclusion

The electrochemical equivalent of copper is obtained (z) = ___________ g/C

Possible Errors

1. Error keeping deflection constant.
2. Error measuring the weight of copper plate.

Viva-Voce

1. What is electrochemical equivalent (ECE)?

   Answer: It is the mass of substance liberated by passing of one coulomb of charge. Chemical equivalent is equal to atomic weight/valency, that is, same as equivalent weight. The copper voltmeter contains copper sulfate solution as electrolyte. During preparation of solution, we also add 0.1% HCl/H_2SO_4 by volume because this furnishes additional ions in the solution and thus increases its conductivity.
2. What is direction of current?

   Answer: We have reversed the direction of the current to reduce the error in the measurement of deflection that arises due to the coil of the galvanometer not exactly in the magnetic meridian. The direction of the current is changed only in a part of the circuit.
3. What is reduction factor?

   Answer: It is the current required to produce a deflection of 45° in tangent galvanometer. Reduction factor decreases with the increase in number of turns in the coil and increases by increasing the radius of the coil.
4. What is electrolysis?

   Answer: When an electric current is passed through a liquid (an electrolyte), decomposition of liquid takes place, it is termed as electrolysis. Electrolyte is a solution that conducts electricity through it.
5. What is the effect of two identical coils?

   Answer: The two identical coils having same axis and carrying current in the same direction are kept apart at distance *r*, to maintain the uniform magnetic field in-between the coils. This is because the rate of decrease of field due to one coil is compensated by the rate of increase of field by the other and hence, the compass needle rotates in uniform magnetic field (Fig. 2). Therefore, 6.Helmholtz galvanometer always functions better than tangent galvanometer.

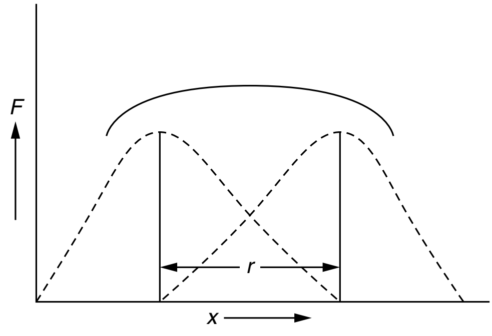


Figure 2

6. What are types of cells?

Answer: Primary cells such as Lechlanché cell, Daniell cell in which the electromotive force is developed due to some chemical reactions taking place in their electrolyte. Lechlanché cell does not give constant current for a long time. The other types are secondary cells where current is passed to convert the electrical energy into chemical energy by electrolysis and after that the chemical energy is converted back into the electrical energy. Lead accumulator is an example of secondary cell.

Experiment 8

Date: ____________

Objective

To verify Stefan's law by electric method.

Apparatus

Stefan's constant apparatus, connection wires and power supply.

Theory and Formula Used

Stefan's law of radiation states that the amount of energy radiated in unit time from the unit surface (E) of a perfectly black body is directly proportional to the fourth power of its absolute temperature (T). Therefore,

$$E = \sigma T^4$$

Where σ is Stefan's constant.

The instrument comprises following built-in parts: (a) Continuously variable, overload and short-circuit protected DC regulated power supply of 0 – 12 V, (b) voltmeter and current meter are mounted on the front panel to measure the voltage and current and (c) one-bulb holder with bulb (12 V) is also mounted on the front panel.

The electric power dissipated from a bulb is taken to be proportional to the *n*th power of the absolute temperature of its filament. The temperature of the filament in turn is directly proportional to the resistance. Hence, by measuring the resistance corresponding to different powers, n can be calculated.

By plotting a graph taking log P along X-axis and log R along Y-axis, the slope of the graph is calculated that gives $\propto$.

$$P = VI \tag{1}$$

where P is the power dissipated from the bulb, V is the voltage across the bulb, I is the current through the bulb and R is the resistance of the filament ($R = V/I$).

Method

Make the connection through patch chords as shown in Fig. 1. Switch on the instrument using ON/OFF toggle switch provided on the front panel. Now increase the DC voltage till the bulb begins to glow. For various values of current, note down the voltage from voltmeter. Calculate the power by using formula

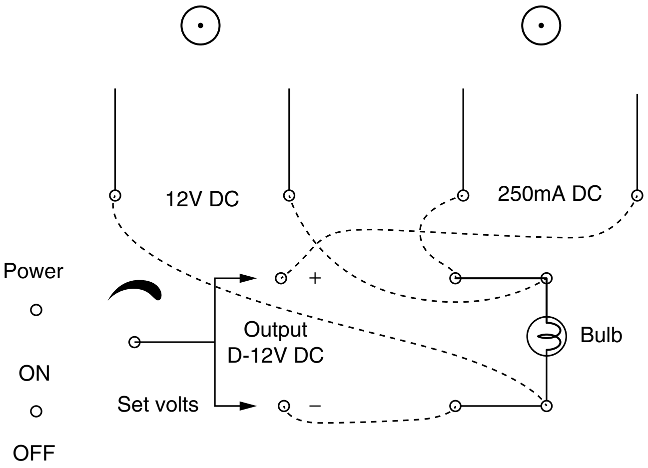


Figure 1

through Eq. (1) and also calculate the filament resistance $R = V/I$, respectively. Plot a graph by taking log P along X-axis and log R along Y-axis, the slope of the graph is calculated that gives the value of $\propto$.

Observations

Sl. No.	*Current I (A)*	*Voltage V (V)*	$P = VI$	*log P*	$R = V/I$	*log R*
1.						
2.						
3.						
4.						
5.						
6.						
7.						
8.						
9.						
10.						
11.						
12.						
13.						
14.						
15.						

Calculation

From the graph the slope of the curve as value of $\propto$ is = ____________

Conclusion

The graph of log P versus log R comes out to be a straight line. Hence, ($P = CT^{\alpha}$) law is verified. Further, the slope of the line $\propto = 4$ and; therefore, the law is verified as a fourth-power law.

Possible Error

1. Error may cause in measuring the voltage and current.

Viva-Voce

1. What is Stefan's law?

   Answer: For a perfectly black body (which absorbs all the incident radiation), the total energy radiated per second by a unit area is directly proportional to the fourth power of the absolute temperature T. However, if the body is not perfectly black, Stefan's law is $E = \sigma\left(T_0^4 - T^4\right)$, where E is the net amount of radiation emitted per unit area by a body at temperature T and surrounded by another body at temp T_0. σ is called Stefan's constant. A similar relation can be held for bodies that are not black $P = C(T^{\propto} - T_0^{\propto})$. For $T \gg T_0$ –the relation can be reduced as $P = CT^{\alpha}$.
2. What is power-radiated P?

   Answer: In this experiment tungsten bulb is used as the radiating body and in the steady state, the electrical power VI should be equal to the radiating power P (neglecting power lost in the leads and through the gas in the bulb).
3. What is Kirchhoff's law?

   Answer: For the given temperature the ratio of the emissive power to the absorptive power (ratio of the radiant energy absorbed by the unit area to the total radiant energy incident on the same unit area of the same surface in the same time) corresponding to a particular wavelength is constant for all bodies and is equal to the emissive power (the radiant energy emitted per second per unit surface area of the body per unit wavelength) of a perfectly black body at the same temperature and wavelength.
4. What is temperature coefficient?

   Answer: The increase in resistance per unit length per unit degree rises in temperature.

Experiment 9

Date: ____________

Objective

To study the variation of magnetic field with distance along the axis of a current carrying coil and then to estimate the radius of the coil.

Apparatus

Stewart and Gee type tangent galvanometer, storage battery, commutator, ammeter, rheostat, one-way plug key, connection wires and a piece of sand paper.

Theory and Formula Used

Stewart and Gee type tangent galvanometer is shown in Fig. 1. It consists of a circular coil of many thin insulated copper wires. It is fixed with its plane vertical on the horizontal bench *AB*. A magnetometer compass box is placed inside the coil such that it can slide on the bench *AB* in such a way that the center of the needle always lies on the axis of the coil. The distance of the needle from the center of the coil can be read on the graduated scale fixed on the arms of the magnetometer.

The variation of magnetic field along the axis of a current carrying circular coil is given by the formula:

$$\frac{2\pi n i r^2}{10(x^2 + r^2)^{3/2}} = H \tan\theta \qquad \text{or} \qquad \frac{2\pi n i r^2}{10(x^2 + r^2)^{3/2}} \propto H \tan\theta$$

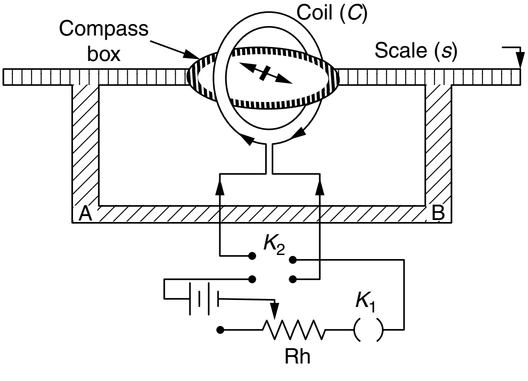


Figure 1

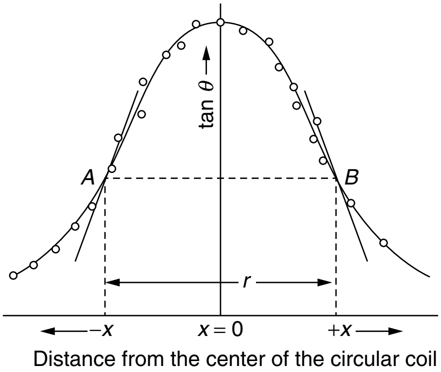


Figure 2

where i is the current flowing through the circular coil of radius r, n is the number of turns in the coil, H is the horizontal component of the Earth's magnetic field and x is the distance on the axis. At center ($x = 0$), the magnetic field due to current carrying coil is maximum that decreases with increase of x. However, H remains same at all points.

Method

Place the magnetometer compass box on the sliding bench so that its magnetic needle is at the center of the coil. By rotating the whole apparatus in the horizontal plane, set the coil in the magnetic meridian roughly. In this case the coil, needle and its image all lie in the same vertical plane. Rotate the compass box till the pointer ends read 0-0 on the circular scale, which is also parallel to the axis of coil. To set the coil exactly in the magnetic meridian setup, see the electrical connections as shown in Fig. 1. Send the current in one direction with the help of commutator and note down the deflection of the needle. Now reverse the direction of the current and again note down the deflection. If the deflections are equal then the coil is in magnetic meridian otherwise turn the apparatus a little, adjust the pointer ends to read 0-0 till these deflections become equal. Using rheostat adjust the current such that the deflection of nearly 70°–75° is produced in the compass needle kept at the center of the coil. Read both the ends of the pointer, reverse the direction of the current and again read the ends of the pointer. The mean of four readings will give the mean deflection at $x = 0$.

Now shift the compass needle through 1 cm, each time along the axis of the coil and note down the mean deflection for each position. Continue this process till the compass box reaches the end of the bench. Repeat the measurements on the other side of the coil exactly in the same manner.

Plot a graph taking x along the X-axis and tan θ along the Y–axis (see Fig. 2). Mark the points of inflection (the points where the curve changes its sign, i.e., from concave to convex) on the curve. The distance between the two points of inflection will be the radius of the coil.

Observations

The radius of the coil (AB) as measured from $X - \tan\theta$ graph = __________ cm
The radius of the coil from the measurements of circumference = circumference/2π = ___________ cm.
The percentage error in the experimental result is calculated by the following formula:

$$\text{Percantage error} = \frac{\text{Standard} \sim \text{Calculated}}{\text{Standard}} \times 100 = \_\_\_\_\_\_\_\_\_\_ \%$$

Sl. No.	***Distance of the Needle from the Center of the Bench (x) cm***	***Deflection on Left Side of the Coil in Degree***					***θ in Degree***	***tan θ***	***Deflection on Right Side of the Coil in Degree***					***θ in Degree***	***tan θ***
		Current One Way		***Current Reversed***		***Mean $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4/4$***			***Current One Way***		***Current Reversed***		***Mean $\theta = \theta_1 + \theta_2 + \theta_3 + \theta_4/4$***		
		θ_1	θ_2	$\theta3$	θ_4				θ_1	θ_2	θ_3	θ_4			
1.	1														
2.	2														
3.	3														
4.	4														
5.	5														
6.	6														
7.	7														
8.	8														
9.	9														
10.	10														
11.	11														
12.	12														

Calculation

Plot the graph between x and $\tan \theta$ and show the radius of coil that is the distance between points of inflection.

Conclusion

The attached graph (see Fig. 2) shows the variation of the magnetic field along the axis of a circular coil carrying current.
The radius of the coil as measured from $X - \tan\theta$ graph = ________ cm

Possible Errors

1. The plane of the coil should be carefully set in the magnetic meridian and the center of the compass box should always lie on the axis of the coil.
2. No magnetic substance or current carrying conductors in the neighborhood of the instrument.

Viva-Voce

1. What is magnetic meridian?

   Answer: A vertical plane passing through the axis of a magnetic needle suspended freely through its center of gravity and rest under Earth's field is called magnetic meridian. To reduce the error arises due to magnetic meridian, we reverse the direction of the current through commutator. Thus, the mean of four readings of deflection will be free from this error.
2. What is the direction of the field?

   Answer: If the face of the coil we look at, the current is flowing in clockwise direction then that face of the coil acts as South Pole. Thus, direction of the field will be away from us.
3. What is the variation of the field along the axis of the coil?

   Answer: It varies as shown in the graph. The points at which the curve changes its direction of curvature are called points of inflection. The distance between them is equal to the radius of the coil. The field is uniform only over a small region at the center. We can get wider region of the uniform field by the use of Helmholtz galvanometer which consists of two exactly similar coils placed coaxially.
4. What is the use of two coils?

   Answer: Any decrease in the intensity of the magnetic field due to one coil is compensated by a corresponding increase in the field due to the other coil, so that the field in the region at the center becomes uniform.
5. What is the direction of current?

   Answer: It is true for any direction of current in the two coils, but current should flow in the two coils in the same direction.
6. What is point of inflection?

   Answer: The point where curve changes its direction of curvature is known as point of inflection and the distance between the two points of inflection is equal to the radius of the circular coil.

Experiment 10

Date: ____________

Objective

To determine the coefficient of viscosity of a liquid by rotating cylinder method.

Apparatus

Rotating cylinder, viscometer (Searle's apparatus), experimental liquid, stop watch, weight box, vernier calipers and thermometer.

Theory and Formula Used

The coefficient of viscosity η is given by

$$\eta = \frac{mgd\left(r_2^2 - r_1^2\right)}{8\pi^2 r_1^2 r_2^2}\frac{T}{l}$$

where r_1 is the radius of the inner cylinder and r_2 is the radius of the outer cylinder, d is the diameter of the drum, l is the length of the inner cylinder immersed in liquid and T is the time period of rotation.

The apparatus consists of two coaxial metal cylinders *A* and *B*, the space between the two cylinders contains the experimental liquid (Fig. 1). The inner solid cylinder *A* is fixed to an axle *E* that is pivoted freely at its ends. The axle can be attached to a drum *M* and to a circular sector *R* by pressing the screw *W*. Two parallel flexible threads (carry two identical scale pans *SS* at the other ends) leave opposite sides of the drum tangentially and passing over two ball-bearing pulleys *PP*. The axle may be rotated under a couple provided by the weights in the scale pans and the number of the rotation of the inner cylinder is determined by observing the transit of an index *I* over the circular sector. The motion is stopped or released by lowering or raising a stop *T* out of a punch hole near the center of the circular sector. The outer cylinder *B* can be raised or lowered with the help of a screw *N*, thus varying the length of the inner cylinder immersed in the liquid that can be seen through a graduated glass window fixed to the outer cylinder.

Method

First measure the radii of the inner and outer cylinders, respectively, r_1 and r_2 as well as the diameter of the drum *d* with the help of vernier calipers. Now pour the experimental liquid whose viscosity is to be measured, into the outer cylinder. Insert the inner solid cylinder in the liquid and tight its ends with the lubricated screws to reduce the friction. The drum is attached to the inner cylinder on which number of

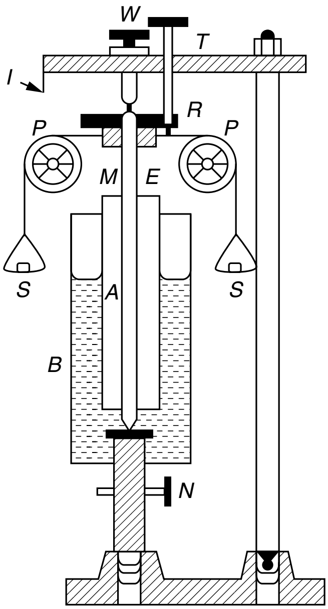


Figure 1

turn of the threads carrying the pans with equal weight are wound round it. The whole system is clamped by introducing the stop into the punch hole. Now count the number of rotation of cylinder and note down the time *t* for unwinding thread for particular length *l* of cylinder immersed in the liquid. The length of the inner cylinder dipped in the liquid can be changed by raising or lowering the outer cylinder. Take several sets of readings for different mass *m* and number of turns of thread over the drum.

Observations

Room temperature = ____________ °C
Radius of the inner cylinder (r_1) = ____________ cm
Radius of the outer cylinder r_2 = ____________ cm
Diameter of the drum (d) = ____________ cm
Mass of the each scale pan (w) = ____________ g

Determination of time periods T_1, T_2 and T_3.

Mass on Each Scale Pan (w_2) *g*	*Total Mass* $m = w_1 + w_2$	*Length of the Inner Cylinder Immersed l (cm)*	*No. of Rotations n*	*Taken Time in second t*	*Time Period* $T = \frac{t}{n}$	*Average Time Period*
0						
0						T_1 =
0						

(*Continued*)

(Continued)

Mass on Each Scale Pan (w_2) g	*Total Mass $m = w_1 + w_2$*	*Length of the Inner Cylinder Immersed l (cm)*	*No. of Rotations n*	*Taken Time in second t*	*Time Period $T = \frac{t}{n}$*	*Average Time Period*
1						
1						$T_2 =$
1						
2						
2						$T_3 =$
2						

Calculation

By substituting the values in the formula, calculate the coefficient of viscosity of liquid separately for each value of T (i.e., T_1, T_2 and T_3) as η_1, η_2 and η_3. By taking mean of these values, we obtain average value of the coefficient of viscosity of given liquid.

Conclusion

The coefficient of viscosity of the given liquid η is found as = ___________ poise

Possible Errors

1. The error may be possible in measuring different radii of the apparatus.
2. It may also occur in measurement of timing when thread just unwound.
3. It is also impossible to get frictionless ends of the inner cylinder.

Viva-Voce

1. What is viscosity?

   Answer: The property of a liquid by virtue of which it opposes the relative motion between its different layers is known as viscosity. In liquids, viscosity arises due to frictional forces between adjacent layers of the liquid when they are in motion.
2. What is coefficient of viscosity?

   Answer: The coefficient of viscosity is defined as the viscous force acting per unit area between two adjacent layers moving with unit velocity gradient. The SI unit of viscosity is N-s/m^2 and CGS unit is dyn-s/cm^2 (poise).
3. What is the effect of temperature and pressure on coefficient of viscosity?

   Answer: The coefficient of viscosity increases with the rise in pressure and decreases with the rise in temperature.

Experiment 11

Date: ____________

Objective

To study the polarization of light using He–Ne laser.

Apparatus

Polarizer, analyzer, photo detector with power meter, mounts, optical bench, beam expander and He–Ne laser.

Theory and Formula Used

Light waves are electromagnetic waves and transverse in nature. Their electric and magnetic field vectors are oscillating and perpendicular to each other and also perpendicular to the direction of propagation. If oscillations of electric vector are confined to a single plane, then it is called a plane of vibration or linearly polarized electromagnetic wave. Ordinary light waves transmitted through polarizer become completely plane polarized light. Malus investigated that when completely plane polarized light is incident on the analyzer, the intensity I of the transmitted light through the analyzer is directly proportional to the square of the cosine of angle between the transmission axis of the analyzer and polarizer, that is,

$$I \propto \cos^2 \theta$$

where θ is the angle between the axis of the analyzer and the polarizer. But we know that amplitude of transmitted light resolves into parallel and perpendicular components. Parallel component ($A_0 \cos \theta$) passed through the analyzer; however, perpendicular ($A_0 \sin \theta$) will be absorbed. Therefore, intensity of transmitted light is:

$$I = (A_0 \cos \theta)^2 = I_0 \cos^2 \theta$$

where I_0 is the maximum intensity of polarized light.

Method

The light coming from He–Ne laser is focused on to a pinhole by lens L_1 and the pinhole is placed at the focus of lens L_2, so that a parallel extended beam falls on the polarizer. Polarizer and analyzer are properly placed in this expended beam (see Fig. 1). The energy of emerging light from analyzer is measured through power meter and detector. Now analyzer is rotated in steps of 5° from the position of transmission direction

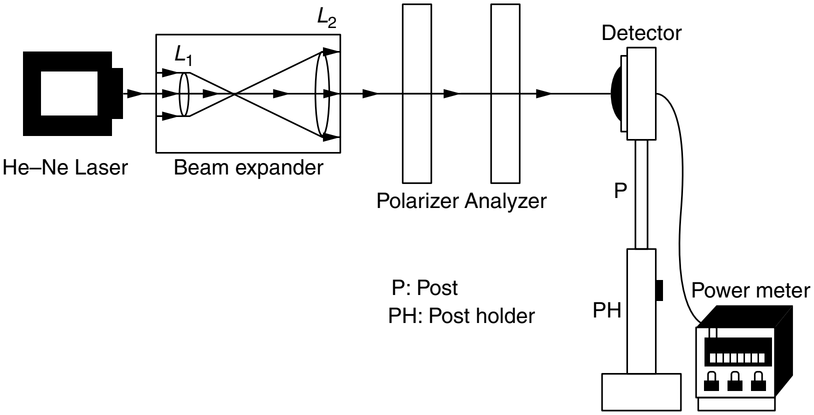


Figure 1

and output of power meter is read for all orientations between 0° and 180°. Again rotate the analyzer to get the maximum intensity, I_0 (If $\theta = 0°$ or $180°$, then $I = I_0 \cos^2 0° = I_0$, i.e., axes of polarizer and analyzer are parallel and if $\theta = 90°$, then $I = I_0 \cos^2 90° = 0$, i.e., axes of polarizer and analyzer are perpendicular.)

Observations

Maximum intensity, $I_0 =$ __________ units.

Sl. No.	*Analyzer Rotation (θ) in Degree*	*Power Meter Reading (I_0) Unit*	*$I_0 \cos^2\theta$*
1.			
2.			
3.			
4.			
5.			
6.			
7.			

Calculation

A graph is plotted between the angle of incident θ on X-axis versus I on Y-axis. The graph will be $\cos^2\theta$.

Conclusion

Thus, Malus law, $I = I_0 \cos^2\theta$ is verified.

Possible Errors

1. The polaroids should be adjusted carefully during observation of the intensity of light.
2. The photodetector should be avoided from stray light effect.

Viva-Voce

1. What is LASER?

   Answer: It stands for Light Amplification by Stimulated Emission of Radiation. It is a device that is used to amplify or generate radiation through stimulated emission process. Laser light is highly monochromatic, highly coherent, non-divergent and intense.
2. What is stimulated emission?

   Answer: It is a process in which the electron is triggered to undergo the transition by the presence of photons of energy ($E_2 - E_1$), the difference of energies between the two transitions level.
3. What is laser action?

   Answer: To produce laser action, one has to create the population inversion using optical pumping.
4. What is He–Ne laser?

   Answer: It is the most widely used laser with continuous power output in the mW range. Due to simple operation, it is widely used in laboratories to perform a number of experiments. In this laser, active medium is a mixture of He and Ne in the ratio of 10:1 or 7:1. Ne provides the energy levels for the laser transitions while He plays an important role in an efficient excitation mechanism for the Ne atoms. He–Ne laser emits red line of wavelength 635 nm.
5. What is angle of polarization or polarizing angle?

   Answer: The certain incidence angle (i_p) for a particular material, at which reflected ray is completely polarized, for example; polarizing angle for glass is 57°.
6. What is Brewster's law?

   Answer: Brewster found that the value of polarizing angle depends upon the refractive index of the refractive medium, that is, mathematically tangent of the polarizing angle i_p is equal to the refractive index of the material $\mu = \tan i_p$.
7. What is photodetector?

   Answer: It is a device that is used to convert light energy into electrical energy. It is basically of three types: photoemissive cell, photovoltaic cell and photoconductive cell. In the experiment, we are using photovoltaic cell as it is cheap and do not require any external emf.

For additional knowledge please see viva-voce in Experiment No. 4.

Experiment 12

Date: ____________

Objective

To determine the wavelength of He–Ne laser source using single slit diffraction.

Apparatus

He–Ne laser, single slit, mounts, meterstick, steel rule, and optical bench.

Theory and Formula Used

Let a parallel beam of He–Ne laser light of wavelength λ be incident normally upon a narrow slit $AB = a$ as shown in Fig 1 (a). The diffraction pattern obtained on the screen consists of a central bright band, having alternate dark and weak bright bands of decreasing intensity on either side of central bright band.

In figure 1 (a),

$$\tan\theta = \sin\theta = \frac{Y}{D}$$

Condition for minimum intensity

$$a\sin\theta = \pm\, n\lambda \text{ and } Y = \pm\, n\, D\frac{\lambda}{a}$$

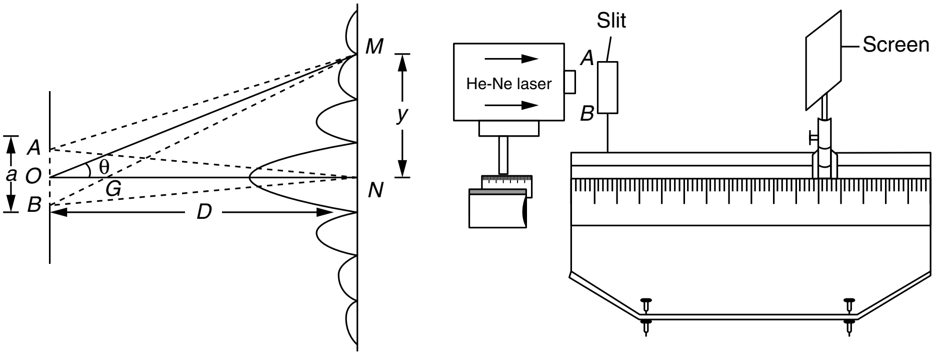
or

$$\lambda = \pm\frac{ay}{nD}$$

Here ± sign indicates on either side to central maximum.

Method

Mount the He–Ne laser and screen on the optical bench as shown in Fig. 1 (b). Now tape a piece of white paper to the screen and adjust the distance between the source and screen in such a way that a tiny circular patch of light is observed in the middle of the screen. After that mount the single slit close to the source, having chosen a slit opening of 0.10 mm and obtain the clear diffraction pattern on the screen. Measure the distance D from the slit to the screen. Mark the location of each of the maxima on either sides of the central



(a) (b)

Figure 1

bright fringe on the paper using pencil and accurately record the positions and widths. The same procedure is repeated by using another piece of paper on the screen and changes the slit opening or another slit with fixed width.

Observations

Sl. No.	*Slit Width a in mm*	*The Distance of Slit from Screen D in m*	*Half Distance Between nth Dark Fringe on the Left and that on the Right of Central Maxima*					*Wavelength λ for Each Slit or Varying Slit Width*	*Mean λ*
			First Dark Fringe Y_1	*Second Dark Fringe* Y_2	*Third Dark Fringe* Y_3	*Fourth Dark Fringe* Y_4	*Mean separation* $Y = \dfrac{Y_1 + Y_2 + Y_3 + Y_4}{1 + 2 + 3 + 4}$		
1.								$\lambda_1 =$	
2.								$\lambda_2 =$	
3.								$\lambda_3 =$	
4.								$\lambda_4 =$	

Calculation

The wavelength of the He–Ne laser is $\lambda = ay/nD =$ ___________ nm

Conclusion

Thus, the wavelength of the He–Ne laser is ___________ nm.

Possible Errors

1. Laser beam cannot be looked directly because it is harmful for your vision.
2. The photodetector should be avoided from stray light effect.
3. The slit width must be measured accurately.
4. All the measurements must be taken carefully to avoid error.

Viva-Voce

1. What is the difference between laser source and filament lamp?

   Answer: Laser beam is highly monochromatic, highly coherent, non-divergent and intense. If a laser beam is focused by a lens or a mirror, its energy is concentrated on a very small area; but filament lamp cannot be concentrated to such a small area.

The additional viva-voce questions are already given in Experiment Nos. 3 and 12 regarding this experiment.

Experiment 13

Date: ____________

Objective

To measure the ballistic constant of a moving coil of a ballistic galvanometer.

Apparatus

Ballistic galvanometer with lamp and scale arrangement, Morse key, one-way key, rheostat, voltmeter, capacitor and a power supply.

Theory and Formula Used

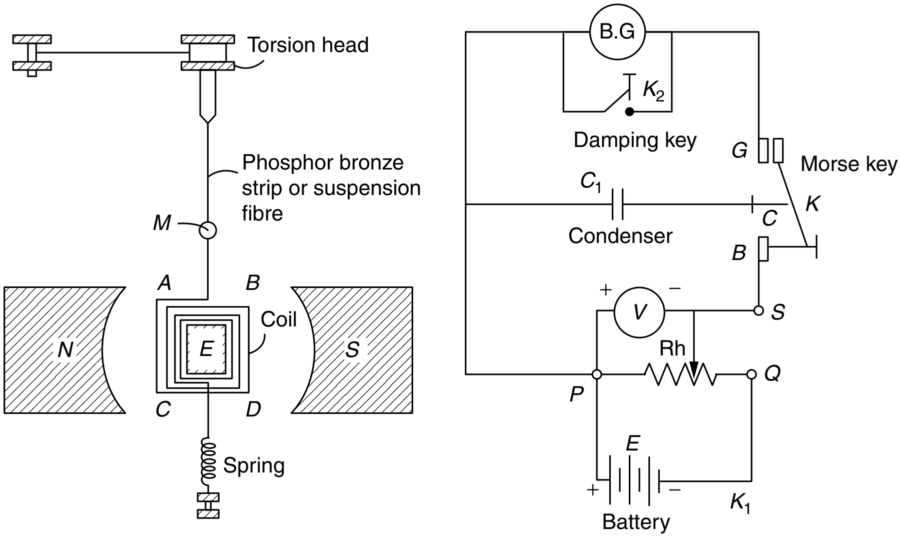
Ballistic galvanometer is designed to measure charge. As shown in Fig. 1 (a), it consists of a rectangular or circular coil *ABCD* of large number of fine insulated copper wire turns wound on a non-conducting frame made of bamboo paper and is suspended between the pole pieces of a permanent horseshoe magnet by means of phosphor–bronze strip. A soft iron cylinder *E* is placed within the coil to make the field radial. The phosphor–bronze strip is attached to a torsional head on the upper side. The lower end of the coil is attached to a spring of a very fine phosphor–bronze wire. A circular mirror *M* is rigidly attached to the phosphor–bronze strip to record the deflection of the coil by lamp and scale arrangement. The charge enters from one end, passes through the suspension wire, coil spring and leaves from the second terminal T_2. The ballistic constant *K* of the ballistic galvanometer is given by

$$K = \frac{CV}{\theta_1\left(1 + \frac{\lambda}{2}\right)}$$

where *C* is the capacity of the condenser in farad, *V* is the voltage of the cell used for charging the condenser in volt, θ_1 is the first observed throw of the ballistic galvanometer, the condenser is discharged through the galvanometer and λ is the logarithmic decrement and is given by

$$\lambda = 0.2303 \log_{10}\left(\frac{\theta_1}{\theta_{11}}\right)$$

CV signifies the quantity of charge passing through the coil of the galvanometer.



(a) (b)

Figure 1

Method

Make the connection as shown in Fig. 1 (b) using rheostat as a potential divider. Any desired voltage (V) can also be applied through this arrangement to condenser C_1 for charging it (voltage should not so high that spot of light as reflected from mirror M goes beyond the limit of scale). Adjust lamp and scale arrangement in such a way sharp light spot is received with pointer wire at the middle of the scale after being reflected from the mirror attached with the suspension wire of the ballistic galvanometer. The Morse key is presented so that the condenser is charged when the capacitor is fully charged (say for about 1/4 min), the Morse key is released so that the condenser discharges itself through the ballistic galvanometer. Note down the first throw (θ_1) and eleventh throw (θ_{11}) of the spot on scale. The same procedure is repeated using another known capacitor but of different capacity (C_2).

Observations

Voltage (V)	*First Throw* (*cm*)		*Eleventh Throw* (*cm*)
	Capacity of Capacitor (μF)		
$V_1 =$	$C_1 =$	θ_1	θ_{11}
	$C_2 =$	θ_1'	θ_{11}'
$V_2 =$	$C_1 =$	θ_1	θ_{11}
	$C_2 =$	θ_1'	θ_{11}'

Calculations

1. The ratio of the capacities of the given capacitor is (θ_1/θ_1') =
2. The ballistic constant of the ballistic galvanometer is given by $K = \dfrac{CV}{\theta_1[1+(\lambda/2)]} =$

   where λ is logarithmic decrement given by $= 0.2303 \log_{10}(\theta_1/\theta_{11})$

Conclusion

1. Thus, the ratio of the given capacitor comes out to be ________
2. The ballistic constant of the ballistic galvanometer is ________ C/cm

Possible Errors

1. Error in counting and measuring the first and eleven throws (note that throws are not counted like the oscillations).
2. For comparing the capacities, the voltage must be same for both the capacitance.

Viva-Voce

1. What is ballistic constant (K)?

   Answer: It is a constant for the galvanometer, which is proportional to the restoring couple per unit twist of the suspended coil and to the time period of oscillation of coil, and inversely proportional to the strength of magnetic field between poles, area of cross section of coil and number of turns on the coil. When K is multiplied by the first throw of the galvanometer, it gives the amount of charge passing through the coil, that is, $q = K\theta$.
2. What is ballistic galvanometer?

   Answer: It is an instrument that is used to measure the electricity or charge, displayed by a varying current of short duration.
3. What is logarithmic decrement?

   Answer: Since the amplitudes of oscillation of the coil of the galvanometer do not remain constant but decrease in each swing because motion of the coil is damped due to the viscosity of air. Therefore, the ratio of the successive amplitudes on either side is called the decrement and the logarithm of this is called the logarithmic decrement.
4. What is a capacitor or condenser?

   Answer: The ability of a condenser to store electrical charge is capacity and equal and opposite charges of the conductors is known as capacitor or condenser. It is directly proportional to the area of the plate, dielectric medium of the two plates and inversely proportional to the distance between two plates. The capacity of the condenser increases by inserting dielectric of higher dielectric constant.
5. What is sensitivity of ballistic galvanometer?

   Answer: The throw in millimeter produced on a scale placed 1 m away from the galvanometer mirror when a current of 1 μA is passed through the coil. Sensitivity can be increased by increasing the area of the coil, magnetic field and the number of turns in the coil.

Experiment 14

Date: ____________

Objective

To measure the high resistance by leakage method.

Apparatus

Ballistic galvanometer with lamp and scale arrangement, a condenser, rheostat, Morse key, a toe key, unknown high resistance, stop watch, plug key and a power supply.

Theory and Formula Used

The high resistance R is determined by the relation

$$\frac{1}{R} = \frac{2.303C}{t}\log_{10}\left(\frac{\alpha}{\theta}\right)$$

where t is the time period of the leakage of condenser through the resistance, C is capacity of the standard condenser, α is the first throw of spot of light when initially the condenser is discharged and θ is the first throw of spot of light when the condenser is discharged through galvanometer after a leakage of charge for time t through R.

Method

Adjust the galvanometer, lamp and scale arrangement and make a potential divider to apply desired voltage to the condenser for charging, etc. This experiment can be divided into two steps. First, the condenser is charged and after self-leakage for time t, discharged through a ballistic galvanometer immediately. Second, the condenser is charged and its charge is allowed to the leak through the unknown resistant for t s and the remaining charge is passed through the ballistic galvanometer.

To perform this experiment, make the connection as shown in Fig 1. To charge the capacitor connect A and C and press the Morse key so that D and F are also connected for a short time (say 15 s), then keeping the Morse key pressed, disconnect A and C of key K_2 and note time for self-leakage on a stop watch. After some measured interval of time, the Morse key is released and the first throw α is recorded. This procedure is repeated several times for different leakage time (t).

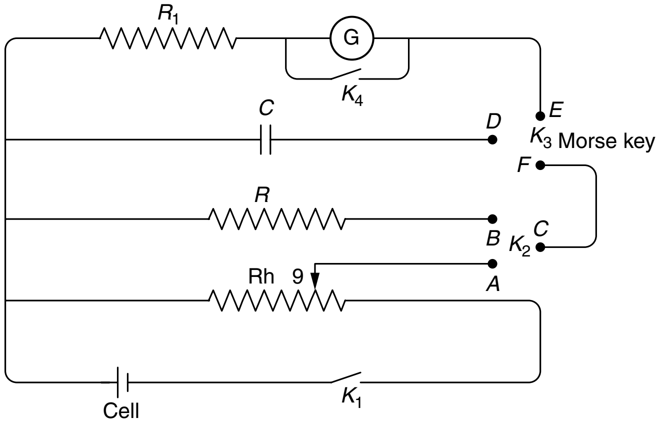


Figure 1

Next after charging the capacitor, it is allowed to leak through unknown high resistance R (by connecting B and C of K_2) for the same time interval (t). After time t, the Morse key is released and the first throw θ is recorded for different values of time.

Observations

Capacity of the condenser $C =$ __________ μF

Sl. No.	*First Throw of Light Spot*		*Leakage Time* (t) *in seconds*	*log*$_{10}$ (α/θ)
	After Self-leakage in cm (α)	*After Leakage Through High Resistance in cm* (θ)		
1.				
2.				
3.				
4.				
5.				

Calculation

The unknown resistance R is determined by the relation

$$\frac{1}{R} = \frac{2.303C}{t}\log_{10}\left(\frac{\alpha}{\theta}\right)$$

Conclusion

The value of the high resistance = ___________ Ω.

Possible Errors

1. Error may occur if time is the same for leakage in both the cases, that is, for self-leakage and leakage through resistance.
2. Error may occur if the condenser with satisfactory rate of leakage is not selected.
3. Error may occur due to leakage through the connecting keys.

Viva-Voce

1. What is leakage method?

   Answer: It is used to determine the order of mega ohm. Low resistance cannot be measured by this method because capacitor will discharge rapidly, hence time cannot be measured accurately.

The additional viva-voce questions are already given in Experiment No. 14 regarding this experience.

Experiment 15

Date: ____________

Objective

To study the Hall effect and to determine Hall coefficient, carrier density and mobility of a given semiconductor material using Hall effect setup.

Apparatus

A rectangular slab of semiconductor crystal of thickness about 0.3 mm, electromagnet, search coil, ballistic galvanometer, millivoltmeter, battery, ammeter, keys and connection wires.

Theory and Formula Used

If a magnetic field is applied perpendicular to the current carrying conductor, then a voltage is developed across the specimen in the direction perpendicular to both the current and magnetic fields. This phenomenon is known as Hall effect and the developed voltage is called Hall voltage and the generated electric field is called Hall field.

To determine Hall coefficient, a thin metallic strip or specimen of millimeter and several centimeter long is placed along X-axis and the magnetic field (H_Z) along Z-axis is applied and current (I_x) is passed along X-axis in the specimen that is adjusted through rheostat. Thus Hall voltage V_H is developed between the points a and b referred in Fig. 1, which is measured through millivoltmeter. That will give the Hall coefficient R_H.

We know that due to potential difference an electric field E_H is created that opposes the drift in electrons in presence of magnetic force.

$$R_H = \frac{V_H}{I_x} \cdot \frac{d}{H_z} \text{ m}^3/\text{C}$$

where V_H in volt, I_x in ampere, d in meter and H_Z in Wb/m^2.

Carrier density

$$n = -\frac{1}{R_H e} \text{ m}^{-3}$$

where $e = 1.6 \times 10^{-19}$ C.

Hall angle

$$\phi = \frac{V_H}{V_X} \cdot \frac{l_x}{b} \text{ rad}$$

where l_x and b both are in meter.

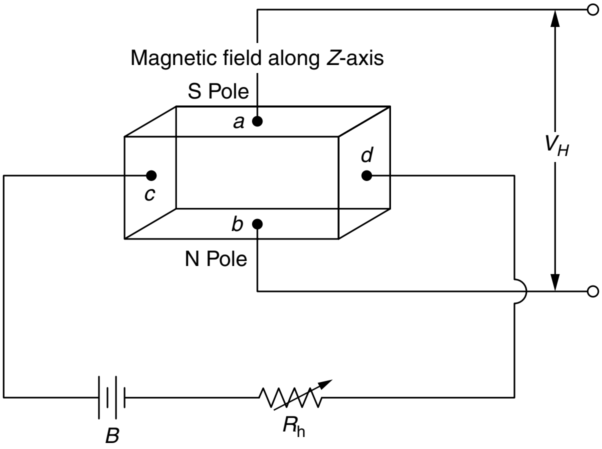


Figure 1

Mobility

$$\mu = \frac{\phi}{H_z} \text{ rad m}^2/\text{Wb}$$

Method

Place the metallic strip in the strong magnetic field produced by the magnet and make other connections according to Fig. 1. A suitable current (I_x) is passed along X-axis in the metallic strip, which is adjusted through rheostat and measured by ammeter. Measure the Hall voltage V_H through millivoltmeter and V_X by voltmeter. Take different value of I_x through rheostat and note corresponding values of V_X and $V_{H.}$ This procedure is repeated several times for different values of I_x. Then plot a graph between V_H versus I_x, which will be straight line whose slope will be $\frac{V_H}{I_x}$. Measure magnetic field H_z with ballistic galvanometer or flux meter.

Observations

Magnetic field H_z = ___________ Wb/m^2
Width of the metallic strip along Z-axis d = ___________ m
Width of the metallic strip along Y-axis b = ___________ m
Length of the metallic strip along X-axis l_x = ___________ m

Measurement for Hall voltage:

Sl. No.	*Current I_x in Ampere*	*Applied Voltage V_X in Volt*	*Hall Voltage V_H in Volt*
1.			
2.			

(*Continued*)

(Continued)

Sl. No.	Current I_x in Ampere	Applied Voltage V_X in Volt	Hall Voltage V_H in Volt
3.			
4.			
5.			
6.			
7.			
8.			
9.			
10.			

Calculations

Plot the graph between V_H on Y-axis and I_x on X-axis that comes out to be a straight line. The slope of the curve gives $\frac{V_H}{I_x}$. Hall coefficient will be

$$R_H = \frac{V_H}{I_x} \cdot \frac{d}{H_z} = \_\_\_\_\_\_\_\_ \text{ m}^3/\text{C}$$

Number of charge carriers per unit volume

$$n = -\frac{1}{R_H e} = \_\_\_\_\_\_\_\_ \text{ m}^{-3}$$

Hall angle

$$\phi = \frac{V_H}{V_X} \cdot \frac{l_x}{b} = \_\_\_\_\_\_\_\_ \text{ rad}$$

Mobility

$$\mu = \frac{\phi}{H_Z} \_\_\_\_\_\_\_\_ \text{ rad m}^2/\text{Wb}$$

Conclusion

Hall coefficient $R_H =$ __________ m^3/C
Number of charge carriers per unit volume $n =$ __________ m^3
Hall angle $\phi =$ __________ rad
Mobility $\mu =$ __________ $\text{rad m}^2/\text{Wb}$

Possible Errors

1. All the measurements should be taken carefully.
2. Magnetic field should be constant for one set of observation.
3. Current should not be large.

Viva-Voce

1. What is Hall potential?

   Answer: In the presence of external magnetic field, a force $F = ev_dB$ acts on electrons to push them toward $+Y$-axis according to Fleming's left-hand rule. So normal to Y-axis of the specimen will become negative and the other side will become positive, thus a potential difference is created, which is Hall potential. It depends upon the nature of charge carriers that decides whether the specimen semiconductor is of *n*-type or *p*-type.
2. Why Hall potential is developed?

   Answer: Due to polarization of charges, an electric field E_H is created that causes Hall potential.
3. What are Hall coefficients?

   Answer: Numerically defined as Hall electric field produced by unit current density and unit magnetic field. Hall coefficient increases with the decrease in number of charge carriers per unit volume.
4. What is mobility?

   Answer: It is the drift velocity of the charge carriers acquired per unit electric field. Mobility of charge carrier increases with the increase of electric conductivity of specimen. Conductivity of the semiconductor is directly proportional to the concentration of charge carriers. The concentration of charge carrier depends on temperature.
5. What is the importance of Hall effect?

   Answer: The sign (positive for holes and negative for electrons) of charge carriers is determined. The number of charge carriers per unit volumes is calculated easily as well as mobility of charge carriers is also measured. It can also be used to determine whether the given material is metal, insulator or semiconductor.
6. What are the applications of Hall effect?

   Answer: Determination of the type of semiconductor, carrier concentration and measurement of magnetic flux density.

Experiment 16

Date: ____________

Objective

To determine the energy band gap of a given semiconductor material.

Apparatus

Power supply (DC 3 V fixed), microammeter, electrically heated oven, thermometer and semiconductor diode.

Theory and Formula Used

In a semiconductor, there is an energy gap between valence band and conduction band and, therefore, for the conduction of electricity a certain amount of energy is to be given to the electrons so that it goes from the valence band to the conduction band. So the energy required is the measure of the energy band gap between the two bands. When a *p–n* junction is reverse biased, the conduction of electricity is due to the minority charge carriers whose concentration is dependent on the energy gap ΔE. The reverse current I_s is the function of the temperature of the junction diode. Its relation may be expressed as

$$\log I_s = \text{const} - 5.036\Delta E\left(\frac{10^3}{T}\right)$$

where T is in Kelvin and ΔE is in eV. A graph is plotted between $\log I_s$ and $10^3/T$ that comes out to be a straight line. The slope of this line would be 5.036 ΔE, which gives the value of band gap

$$E_g = \frac{\text{Slope}}{5.036}\,\text{eV}$$

Method

Connect each component of the apparatus according to the circuit diagram as shown in Fig. 1. Put the diode in place on the board for heating and the thermometer inserted to measure the temperature. Start heating by connecting ovens lead to mains up to 65°C temperature. As soon as the temperature reaches about 65°C, switch off the oven. Temperature further rises and becomes stable at about 70°C. When the temperature of the oven begins to fall, take observations of current I_s and temperature T in the steps of 5 μA till the temperature falls from its stable value to room temperature.

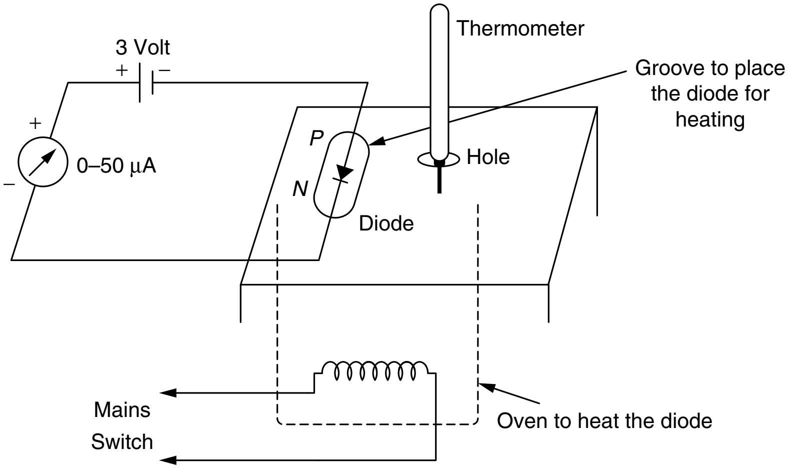


Figure 1

Observations

Sl. No.	*Reverse Saturation Current I_s (μA)*	*log I_s*	*Temperature of the Oven T (°C)*	*Temperature T (K)*	*$10^3/T$*
1.					
2.					
3.					
4.					
5.					
6.					
7.					
8.					
9.					
10.					

Calculation

Plot a graph between log I_s along *Y*-axis and $10^3/T$ along *X*-axis and find slope *AB*/*BC*. Hence

$$E_g = \frac{\text{Slope}}{5.036} = \underline{\hspace{3cm}}\ \text{eV}$$

Conclusion

The energy band gap of the semiconductor material (Ge) = __________ eV.

Possible Errors

1. Thermometer should be properly inserted to take the accurate measurement of temperature.
2. Diode should be placed in proper way so that it is in good contact of the heat of the oven.
3. Maximum temperature should not exceed 80°C.

Viva-Voce

1. What is energy gap?

   Answer: It is the gap between valence band and conduction band in semiconductor. For Ge, it is 0.7 eV.

2. What are valence band and conduction band?

   Answer: The electrons in the outermost orbit of an atom form valence band by a series of close energy levels. Valence band may be partially or completely filled. The higher permitted band is the conduction band and the electron bands occupying this band are called conduction electrons.

3. How semiconductor behaves as an insulator at 0°K?

   Answer: At 0°K, electron in the valence band does not have sufficient energy to cross the energy gap to reach conduction band and thus it behaves as an insulator.

Experiment 17

Date: ____________

Objective

To calibrate the given ammeter and voltmeter by potentiometer.

Apparatus

A potentiometer, storage battery, rheostat, main battery, a standard cell, galvanometer, shunt, high resistance, standard 1-Ω resistance, given ammeter and voltmeter, a two-way key, three single-way keys and connection wires.

Theory and Formula Used

For calibration of voltmeter, we connect the voltmeter in parallel with the standard 1-Ω resistance. In this case, the errors in the voltmeter readings are obtained by following relations:

$$V' - V = \left(\frac{El_2}{l_1} \right) - V$$

where V is the potential difference between two points read by voltmeter, V' is the potential difference between the same two points read by potentiometer, E is the emf of the standard cell, l_1 is the length of the potentiometer wire corresponding to the emf of standard cell and l_2 is the length of the potentiometer wire corresponding to the potential difference (V') measured by potentiometer.

Method

First, complete the circuit with storage battery, a plug key, and a standard 1-Ω resistance all in series and the given voltmeter across the standard resistance according to Fig. 1. To test the correctness of the circuit, place the jockey near the point A and after that at other point B of the potentiometer wire. If the deflection is in opposite direction, the connections are correct. For the standardization of the potentiometer wire, close K_1 and K_2(i), place the jockey on the last end B of the tenth wire, and observe the deflection in galvanometer, which is reduced to zero through rheostat in the main circuit. Note down the total balancing length l_1, that is, 1000 cm for 10 wires and emf of standard cell and calculate potential gradient using $k = E/l_1$. For the calibration of voltmeter, open K_2(i) and close K_2(ii) and K_3, adjust the variable point P of the potential

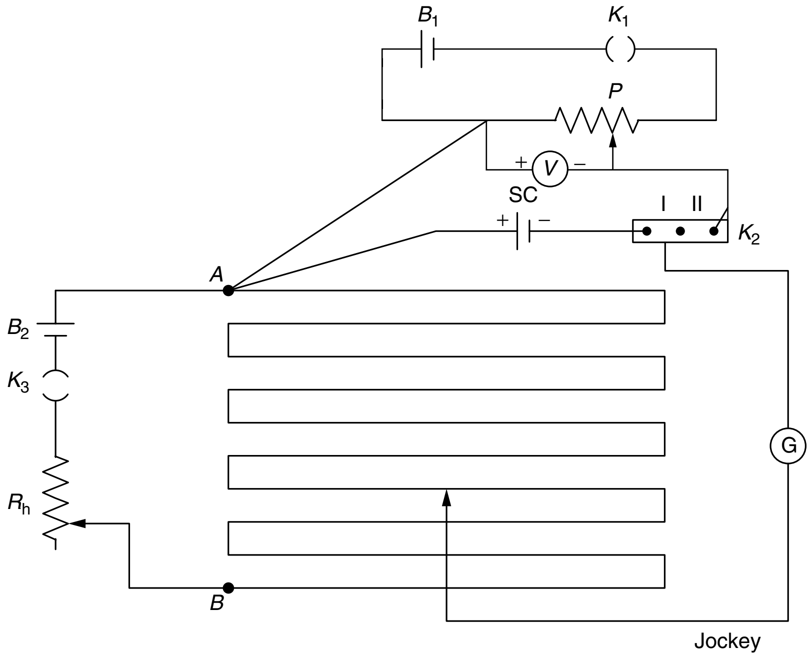


Figure 1

divider so that 0.1-V deflection is observed in voltmeter. Adjust the jockey on the potentiometer wire to find out the null point (*there is no deflection in the galvanometer*) and note down the total balancing length l_2 (*the distance of the potentiometer wire between jockey and zero end A*). Similarly by changing the position of the variable point P take several observations in regular steps of above procedure till the entire range of the voltmeter is covered (i.e., 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 ________ V) and note down the corresponding l_2 in Table B.

Observations

Table for the calibration of potentiometer wire:

Length of the Potentiometer Wire Corresponding to emf of Standard cell, l_1 in cm	
1000	Emf of standard cell $E = 1.080$ V for Daniell cell Potential gradient $k = E/l_1 =$ ______ V/cm

Table for the calibration of voltmeter:

Sl. No.	*Voltmeter Readings in volt V*	*Balancing Length of the Potentiometer Wire l_2 in cm*			$V' = \frac{El_2}{l_1}$	*$(V'-V)$ in Volt*
		No. of Complete Wires n	*Length on Sliding Wire l**	*Total $l_2 = (n \times 100) + l$*		
1.	0.1					
2.	0.2					
3.	0.3					
4.	0.4					
5.	0.5					
6.	0.6					
7.	0.7					
8.	0.8					
9.	0.9					
10.	1.0					

**Length of even number of wires is recorded after subtracting it from 100.*

Calculations

Calculate potential gradient $k = E/l_1 =$ ______ V/cm and after that $V' = El_2/l_1$ for 0.1 to 1.0 V.
Plot a graph between the error $V' - V$ and the voltmeter reading which will be zig-zag in shape.

Conclusion

Thus, the graph obtained by plotting the error versus voltmeter reading is the calibration curve of the given voltmeter.

Possible Errors

1. Connections should be made carefully.
2. Storage battery should be fully charged.
3. The emf of the cell used in the primary circuit should be greater than that of the emf of standard cell.
4. The voltmeter should be calibrated over its entire range.

Viva-Voce

1. What is potentiometer?

   Answer: It is used for measuring potential difference. A 10- or 12-wire potentiometer consists of uniform resistance wire of material (constantan or manganin) that has high specific resistance and low temperature coefficient through which a steady current is maintained by a constant source of emf. The wire should be of uniform thickness so that its resistance per unit length is uniform throughout the length of the wire. Potentiometer is capable of measuring even the smallest potential difference (*through increase the length of potentiometer wire or reduced the current in main circuit using rheostat*), so it is called sensitive.

2. What is zero deflection?

   Answer: It means the reading in galvanometer is zero, that is, emf to be measured has been balanced by the potential difference across that length of the wire.

3. What is potential gradient?

   Answer: The fall of potential per unit length of the potentiometer wire is called potential gradient. It can be changed by changing the emf of the cell or the resistance used with potentiometer wire. It is constant for calibration of ammeter and voltmeter and measurement of thermo-emf.

4. What is calibration of voltmeter?

   Answer: Calibration of voltmeter means checking up the accuracy of its reading. It is necessary to calibrate a voltmeter because mechanical defect and constant use of voltmeter make it erratic. Voltmeter cannot measure emf accurately because some current, howsoever small, will be drawn from the cell. Therefore, we use potentiometer because no current draws at null point and it can be made more sensitive that is why we calibrate voltmeter with the help potentiometer.

5. What is calibration curve?

   Answer: It is not a smooth curve, but a point-to-point curve that indicates the correction to be applied to any of the observed voltmeter readings.

Experiment 18

Date: ____________

Objective

To calibrate the given ammeter by potentiometer.

Apparatus

A potentiometer, storage battery, rheostat, main battery, a standard cell, galvanometer, shunt, high resistance, standard 1-Ω resistance, given ammeter and voltmeter, a two-way key, three single-way keys and connection wires.

Theory and Formula Used

For calibration of ammeter, we connect the ammeter in series with the standard 1-Ω resistance. In this case, the errors in the ammeter readings are obtained by following relation:

$$I' - I = \left(\frac{El_2}{l_1}\right) - I$$

where $I' = V'$ is the current equal to voltage read by potentiometer, I is the current read by ammeter, E is the emf of the standard cell, l_1 is the length of the potentiometer wire corresponding to emf of standard cell and l_2 is the length of the potentiometer wire corresponding to the potential difference (I') measured by potentiometer.

Method

First, complete the circuit with storage battery, a plug key, a standard 1-Ω resistance and the given ammeter all in series according to Fig. 1. To test the correctness of the circuit, place the jockey near the point A and after that at other point B of the potentiometer wire. If the deflection is in opposite direction, the connections are correct. For the standardization of the potentiometer wire, close K_1 and K_2(i), place the jockey on the last end B of the tenth wire, and observe the deflection in galvanometer, which is reduced to zero through rheostat in the main circuit. Note down the total balancing length l_1, that is, 1000 cm for 10 wires and emf of standard cell and calculate potential gradient using $k = E/l_1$. For the calibration of voltmeter, open K_2(i) and close K_2(ii) and K_3, adjust the variable point P of the potential divider so that 0.1-A

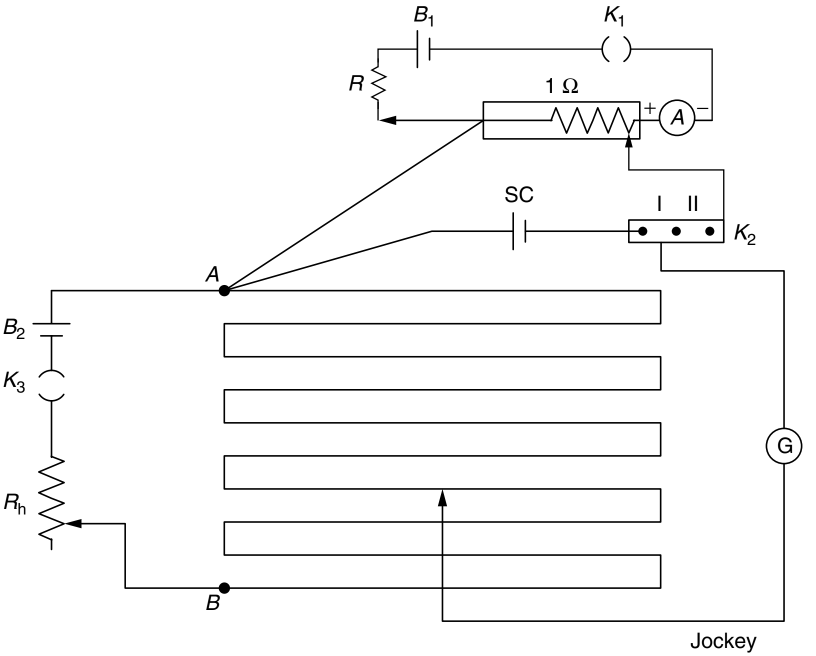


Figure 1

deflection is observed in ammeter. Adjust the jockey on the potentiometer wire to find out the null point (*there is no deflection in the galvanometer*) and note down the total balancing length l_2 (*the distance of the potentiometer wire between jockey and zero end A*). Similarly by changing the position of the variable point P take several observation in regular steps of above procedure till the entire range of the ammeter is covered (i.e., 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8 ________ A) and note down the corresponding l_2 in Table B.

Observations

Table for the calibration of potentiometer wire:

Length of the Potentiometer Wire Corresponding to emf of Standard Cell, l_1 in cm	
1000	Emf of standard cell $E = 1.080$ V for Daniell cell Potential gradient $k = E/l_1 =$ _______ V/cm

Table for the calibration of voltmeter:

Sl. No.	*Ammeter Readings in Ampere I*	*Balancing Length of the Potentiometer Wire l_2 in cm*			$I' = \frac{El_2}{l_1}$	*($I' - I$) in Ampere*
		No. of Complete Wires n	*Length on Sliding Wire l**	*Total* $l_2 = (n \times 100) + l$		
1.	0.1					
2.	0.2					
3.	0.3					
4.	0.4					
5.	0.5					
6.	0.6					
7.	0.7					
8.	0.8					
9.	1.9					
10.	1.0					

**Length of even number of wires is recorded after subtracting it from 100.*

Calculations

Calculate potential gradient $k = E/l_1 =$ _______ V/cm and after that $I' = El_2 / l_1$ for 0.1–1.0 A. Plot a graph between the errors ($I' - I$) and the ammeter reading which will be zig-zag in shape.

Conclusion

Thus, the graph obtained by plotting the error versus voltmeter reading is the calibration curve of the given ammeter.

Possible Errors

1. Connections should be made carefully.
2. Storage battery should be fully charged.
3. The emf of the cell used in the primary circuit should be greater than that of the emf of standard cell.
4. The voltmeter should be calibrated over its entire range.

Viva-Voce

1. What is standard cell?

   Answer: Here we are using Daniell cell having emf 1.08 V, but cadmium cell (emf 1.0186 V) is always better than Daniell cell because its emf does not vary appreciably with the temperature and is able to maintain a steady current.

Experiment 19

Date: ____________

Objective

To measure attenuation in an optical fiber.

Apparatus

He–Ne laser, 20X microscope objective, fiber chuk, optical fiber, photodetector and digital multimeter.

Theory and Formula Used

Attenuation or losses in the fiber are due to absorption, scattering bending and microbending. Therefore, loss in dB/km is given by

$$\text{Loss} = \frac{10}{z}\log\frac{P_0}{P_z}\ \text{dB/km}$$

where P_0 and P_z are the power at the input and output end of the fiber, respectively, and z is the length between input and output ends of the fiber whose attenuation is to be calculated.

Method

First, complete the circuit according to Fig. 1. After that measure the length z of the fiber and mount both ends of the fiber on fiber chuck. Focus the light from the laser into fiber input end using 20X microscope objective.

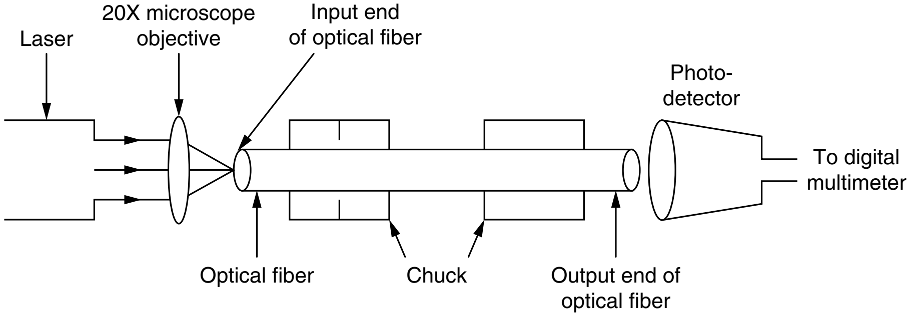


Figure 1

Set the multimeter in DC voltage mode and connect it to photodetector. Then place the photodetector at the input end of the fiber in such a way that light from the microscope objective falls on it. Note the multimeter reading, that is, V_1. Now place the photodetector at output end of the fiber in such a way that light from microscope objective falls on it. Again, note the multimeter reading, that is, V_2.

Observations

Length of optical fiber, $z =$ ________ km

Sl. No.	*Measurement of V_1 at Input End*	*Measurement of V_2 at Output End*	*Loss*

Calculation

Since, $P \propto V^2$, $P_0 \propto V_1^2$ and $P_z \propto V_2^2$,

$$\text{Loss} = \frac{10}{z}\log\frac{P_0}{P_z}$$

or

$$\text{Loss} = \frac{20}{z}\log\frac{V_1}{V_2} = \_\_\_\_\_\_\_\_ \text{ dB/km}$$

Conclusion

Thus, the attenuation or losses in optical fiber is…dB/km.

Possible Errors

1. Connections should be made carefully.
2. Eyes should be protected from the laser beam.
3. Take reading carefully.

Viva-Voce

1. What do you understand by fibre optics?

   Answer: Fibre optics is made of glass or plastic pipes as thin as human hair, through which light can be propagated. It consists of essentially three regions. The innermost region is known as core. It is made of glass or plastic. The middle region which covers the core with another layer of glass or plastic is called the cladding. The outermost region is called Jacket. It is also made of plastic or polymers.
2. What is the principle of fibre optics?

   Answer: Fibre optics based on the principle of total internal reflection (TIR) so that optical signal can travel long distances with negligible loss.
3. Give the necessary conditions for total internal reflection.

   Answer: The necessary conditions for total internal reflection are

   1. Light wave should be propagated from denser medium to rarer.
   2. The angle of incident should be greater than that of the angle of incidence.
4. What is acceptance angle and numerical aperture?

   Answer: The maximum angle at which light entering the core is transmitted through the fibre and propagated without refracted is known as acceptance angle. The sin of acceptance angle is known as numerical aperture.
5. How is light transmitted through fibre optics?

   Answer: When the light signal is incident on the core and makes an angle of incidence greater than that of the critical angle of the core-cladding surface, multiple total internal reflection phenomena occurs. In this way, light signals can be propagated or transmitted through the fibre.
6. What do you understand by modes of propagation?

   Answer: When light ray travel along different ray paths in fibre is known as modes of propagation.
7. What is attenuation?

   Answer: Attenuation or losses in the fiber are due to absorption, scattering bending and microbending of light signal.

B.Tech
(Sem. I) Odd Semester Theory Examination 2013–14
Engineering Physics–I

Time: 2 Hours **Total Marks: 50**

Note: There are three Sections A, B and C in this paper. Questions are to be done from all three Sections.

SECTION–A

1. Attempt all parts. Give answer of each part in short: **(2 × 5 = 10)**

(a) What do you understand by time dilation?

Answer: A moving clock appears to go slow by a factor of $\sqrt{1-\frac{v^2}{c^2}}$. This is time dilation.

(b) What are mass less particles?

Answer: A particle which has zero mass is known as mass less particle.

$$E = \sqrt{(m_0c^2)^2 + (pc)^2}$$

Velocity of mass less particle is same as velocity of light ($v = c$).

(c) In Newton's ring experiment fringe width decreases with the increase of order of fringe. Explain why?

Answer: $\beta = \frac{\lambda}{2\mu\theta}$ shows that when the wedge angle θ gradually increases, the fringe width β decreases.

(d) How the unpolarized light and circularly polarized light distinguish?

Answer: For detection of circularly polarized light, it is made to fall on rotating Nicol. If the intensity of emergent light remains uniform, then emerging light is either circularly polarized or unpolarized.

To differentiate between the two, the beam is allowed to be incident on a quarter wave plate and then on a rotating Nicol. If the variation in intensity is observed from zero to maximum, the emerging light is circularly polarized, on the other hand, if no variation in intensity is observed, the light is unpolarized.

(e) What do you mean by population inversion?

Answer: If the number of atoms are more in excited state in comparison to ground state, then this condition is known as population inversion.

SECTION–B

2. Attempt any three parts. All parts carry equal marks: **(5 × 3 = 15)**

(a) Calculate the length of one meter rod moving parallel to its length when its mass is 1.5 times of its rest mass.

Answer: We have

$$m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}},\ 1.5m_o = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}},\ \sqrt{1-\frac{v^2}{c^2}} = 2/3$$

So

$$l = l_o\sqrt{1-\frac{v^2}{c^2}} = 1\times\frac{2}{3} = 0.67\,\text{m}$$

(b) The speed of electron is measured to 5.0×10^3 m/s to an accuracy of 0.003%. Find the uncertainty in determining the position of the electron (mass of electron is 9.1×10^{-31} kg and Planck's constant is 6.62×10^{-34} J-sec.)

Answer: We have $\Delta v = 0.003 \times 5 \times 10^3/100 = 0.15$ m/s, $h = 6.625 \times 10^{-34}$, $\Delta p \times \Delta x = \hbar$, Now

$$\Delta x = \frac{h}{2\pi m\Delta v} = 7.72\times10^{-4}\,\text{m}$$

(c) Newton's rings are observed in reflected light of wavelength 6000 Å. The diameter of 10th dark ring is 0.5 cm. Find the radius of curvature of the lens and thickness of air film.

Answer: We have $D_n = 0.5$ cm, $\lambda = 6000$ Å, $n = 10$. Now

$$D_n\sqrt{4n\lambda R},\quad R = \frac{D_n^2}{4n\lambda} = 104.16\,\text{cm}$$

$$2t = \frac{D_n^2}{8R} = 3\times10^{-4}\,\text{cm}$$

(d) A diffraction grating used at normal incidence gives a yellow line ($\lambda = 6000$ Å) in a certain spectral order superimposed on a blue line ($\lambda = 4800$ Å) of next higher order. If the angle of diffraction is 60°, calculate grating element.

$$\lambda_1 = 6000\,\text{Å},\ \lambda_1 = 4800\,\text{Å},\ \theta = \sin^{-1}\left(\frac{3}{4}\right)$$

Answer:

$$(a+b)\sin\theta = n\lambda_1$$
$$(a+b)\sin\theta = (n+1)\lambda_2$$

So

$$n = \frac{\lambda_2}{\lambda_1-\lambda_2}$$

$$a+b = \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)\sin\theta} = 3.2\times10^{-4}\,\text{cm}$$

(e) The refractive indices of quartz for polarized light μ_e and μ_o are 1.5508 and 1.5418, respectively. Calculate phase retardation for $\lambda = 5000$ Å when the plate thickness is 0.032 mm.

Answer: We have $\mu_e = 1.5508$, $\mu_o = 1.5418$, $t = 0.032$ mm and $\lambda = 5000$ Å.

$$\text{Phase retardation } = \frac{2\pi}{\lambda} \times \text{path difference} = \frac{2\pi}{\lambda} \times (\mu_o - \mu_e) \times t = 3.617 \text{ rad}$$

SECTION–C

Note: Attempt all questions in this section. All questions carry equal marks.

3. Attempt any one part of the following: **(5 × 1 = 5)**

(a) State Einstein's postulates of special theory of relativity. Explain why Galilean relativity failed to explain actual result of Michelson–Morley Experiment.

Answer: Einstein's postulates:

1. Laws of physics are same for all inertial frames of reference.
2. The speed of light is same for all inertial frames of reference.

According to Galilean transformation equation, length of the object remains unchanged in moving frame and velocity of light becomes $c - v$ or $c + v$ in the moving frame. This was the main reason for failure of relativity theory.

For more information see Chapter 1, Secs. 1.7 and 1.8.1

(b) Show that the relativistic invariance of the law of conservation of momentum leads to the concept of variation of mass with velocity and equivalence of mass and energy.

Answer: Proof of variation of mass with velocity. For more information see Chapter 1, Sec. 1.14.

4. Attempt any one part of the following: **(5 × 1 = 5)**

(a) Deduce relation between phase and group velocity in a medium where velocity is frequency dependent. What happens if the phase velocity is independent of frequency?

Answer: Phase velocity $(v_p) = \frac{\omega}{k}$, $\omega = v_p k$

$$\text{Group velocity } (v_g) = \frac{d\omega}{dk} = \frac{d(v_p k)}{dk} = v_p - \lambda \frac{dv_p}{d\lambda}$$

If phase velocity is frequency independent then

$$\frac{dv_p}{d\lambda} = 0, \; v_g = v_p$$

For more information see Chapter 2, Sec. 2.9.3.

(b) A particle of mass m is confined to a one-dimensional box of length L. Derive an expression for wave function and energy.

Answer:

$$\psi = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} L\right)$$

$$E_n = \frac{n^2 \pi^2 h^2}{8mL^2}$$

For more information see Chapter 2, Sec. 2.17.2.

5. Attempt any one part of the following: **(5 × 1 = 5)**

(a) Discuss the interference in thin film in reflected light. What happens when film is excess thin?

Answer: We have

Path difference = $2\mu t \cos r$

Effective path difference = $2\mu t \cos r - \lambda/2$

For constructive interference

$$2\mu t \cos r = (2n + 1)\ \lambda/2$$

For destructive interference

$$2\mu t \cos r = n\lambda$$

If film is thin then destructive interference will take place. For more information see Chapter 3, Sec. 3.8.1.

(c) Explain the diffraction pattern obtained with diffraction at single slit. By what fraction the intensity of second maximum reduced from principle maximum?

Answer: Single slit diffraction intensity of secondary maximum is $4/9\pi^2$ of principal maximum. This is nearly 4.5%. For more information see Chapter 4, Sec. 4.4.

6. Attempt any one part of the following: **(5 × 1 = 5)**

(a) What is diffraction grating? Show that its dispersive power can be expressed as $\dfrac{1}{\sqrt{\left(\dfrac{e+d}{n}\right)^2 - \lambda^2}}$, where all the terms have their usual meanings.

Answer: An arrangement consisting of large number of parallel slits of equal width and separated from each other by equal opaque spaces is called a 'diffraction grating'.

Dispersive power derivation $\dfrac{d\theta}{d\lambda} = \dfrac{n}{(\text{a}+b)\cos\theta} = \dfrac{n}{\sqrt{(a+b)^2 - n^2\lambda^2}}$

For more information see Chapter 4, Secs. 4.7 and 4.8.

(b) What do you mean double refraction? Explain the working principle of Nicol prism.

Answer: Double refraction is splitting of ray of light into two rays – ordinary and extra ordinary.

For construction and working of Nicol prism see Chapter 5, Secs. 5.3 and 5.5.

7. Attempt any one part of the following: **(5 × 1 = 5)**

(a) Show that two level pumping scheme has no practical significance for lasing. Explain the principle of three level lasers.

Answer: See Chapter 6, Sec. 6.8.

(b) Discuss different types of optical fiber. Why graded index fibre is better than multimode step index fibre?

For types of optical fiber see Chapter 7, Sec 7.5.
Difference: In graded index fibre multipath time dispersion is zero.

B.Tech
(Sem. II) Examination 2013–14
Engineering Physics–II

Time: 2 Hours **Total Marks: 50**

Note: Attempt questions from all sections. Take standard data as and when required.

SECTION–A

1. Attempt all parts. Give answer of each part in short: **(2 × 5 = 10)**

(a) What is primitive cell?

Answer: The smallest volume that contains the full pattern is called a unit cell. If a unit cell contains lattice points only at its corners, then it is called a primitive unit cell or simple unit cell.

(b) What are the important facts of Laue experiment for X-ray diffraction?

Answer: The important points are

(a) The X-rays are electromagnetic radiations of very short wavelength.

(b) The atoms in a crystal are arranged in a regular three-dimensional array.

(c) What is dielectric loss?

Answer: The amount of energy dissipated in the form of heat by a dielectric medium under the action of external electric field is known as dielectric loss.

(d) What do you understand by Poynting vector?

Answer: Poynting Vector is defined as the energy transported by wave per unit area per unit time. It is denoted by a vector P and can be expressed by the cross product of electric and magnetic fields.

(e) What is Fermi level?

Answer: The highest energy that an electron can have in a conductor at absolute zero temperature.

SECTION–B

2. Attempt any three parts of the following: **(5 × 3 = 15)**

(a) Determine the inter-planer spacing of a lattice plane in a simple cubic lattice with edge 2 Å which cuts the axes in intercepts ratio 3:4:5.

Answer: We have

$$h:k:l = \frac{1}{p}:\frac{1}{q}:\frac{1}{r} = \frac{1}{3}:\frac{1}{4}:\frac{1}{5} = 20:15:12 \quad a = 2\ \text{Å}$$

$$d_{hkl} = \frac{a}{\sqrt{h^2+k^2+l^2}} = \frac{2}{\sqrt{(20)^2+(15)^2+(12)^2}} = 0.07212 \quad \text{Å}$$

(b) A magnetic material has a magnetization of 3000 A/m and flux density 0.005 weber/m^2. Determine the magnetizing force and the relative permeability of material.

Answer: We have $I = 3000$ A/m, $B = 0.005$ Wb/m^2. Now

$$B = \mu_0\,(H + I)$$

So

$$H = \frac{B}{\mu_0} - I = \frac{0.005}{4\pi \times 10^{-7}} - 3000 = 980.89$$

$$\mu = \frac{B}{H} = \frac{0.005}{980.89} = 0.51 \times 10^{-5}$$

$$\mu = \frac{\mu}{\mu_0} = \frac{0.51 \times 10^{-5}}{4\pi \times 10^{-7}} = 4.06$$

(c) If a NaCl crystal is subjected to an electric field of 1500 V/m and the resulting polarization is 4.3×10^{-8} C/m^2, calculate the relative permittivity.

Answer: We have $E = 1500$ V/m, $P = 4.3 \times 10^{-8}$ C/m^2. Now

$$P = \varepsilon_0\,(\varepsilon_r - 1)$$

$$\varepsilon_r = 1 + \frac{P}{\varepsilon_0 E} = 1 + \frac{4.3 \times 10^{-8}}{8.85 \times 10^{-12}} = 1 + 3.24 = 4.24$$

(d) A lamp radiates 500 watt power uniformly in all directions. Calculate the electric and magnetic field intensities at 1 m distance from the lamp.

Answer: We have $P = 500$ Watt, $r = 1$ m. Now

$$S = \frac{P}{4\pi r^2} = 39.80$$

$$EH = 39.80,\ \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7$$

$$\Rightarrow E = 122.44\ \text{V/m},\ H = 0.3250\ \text{A/m}$$

(e) In a sample of intrinsic germanium at room temperature, the mobility of electrons and holes is 0.40 m^2/V-sec and 0.23 m^2/V-sec, respectively. If the electron and hole densities are each equal to 1.5×10^{23} m^{-3}. Find out the electrical conductivity and resistivity of germanium.

Answer: We have $n = p = 1.5 \times 10^{23}\text{m}^{-3}$. The conductivity of intrinsic semiconductor is given by

$$\sigma = e\left(\mu_n + \mu_p\right) n_i = 1.6 \times 10^{-19} \times (0.40 + 0.23) \times 1.5 \times 10^{23} = 1.512 \times 10^4\,\Omega\ \text{m}^{-1}$$

Resistivity of intrinsic semiconductor is given by

$$\rho = \frac{1}{\sigma} = \frac{1}{1.512\times 10^{4}} = 0.66\times 10^{-4}\,\Omega^{-1}\text{ m}$$

SECTION–C

Note: Attempt all questions in this section. All questions carry equal marks. **(5 × 5 = 25)**

3. Attempt any one part of the following: **(1 × 5 = 5)**

(a) What is crystal structure? Explain its type.

Answer: Lattice and basis together forms crystal structure. There are seven types of crystal structures.

Lattice System	*Possible*	*Axial Distances*	*Axial Angles*	*Examples*
Cubic	Simple, bcc, fcc	$a = b = c$	$\alpha = \beta = \gamma = 90°$	NaCl, Zinc Blende, Cu
Tetragonal	Simple, bcc,	$a = b \neq c$	$\alpha = \beta = \gamma = 90°$	White tin, SnO_2, TiO_2, $CaSO_4$
Orthorhombic	Simple, bcc, fcc, base	$a \neq b \neq c$	$\alpha = \beta = \gamma = 90°$	Rhombic sulphur, KNO_3, $BaSO_4$
Hexagonal	Simple,	$a = b \neq c$	$\alpha = \beta = 90°$, $\gamma = 120°$	Graphite, ZnO, CdS
Rhombohedral	Simple,	$a = b = c$	$\alpha = \beta = \gamma \neq 90°$	Calcite ($CaCO_3$), Cinnabar (HgS)
Monoclinic	Simple, bcc	$a \neq b \neq c$	$\alpha = \gamma = 90°$, $\beta \neq 90°$	Monoclinic sulphur, $Na_2SO_4.10H_2O$
Triclinic	Simple,	$a \neq b \neq c$	$\alpha \neq \beta \neq \gamma \neq 90°$	$K_2Cr_2O_7$, $CuSO_4.5H_2O$, H_3BO_3

See Chapter 8, Sec. 8.7.

(b) What are miller indices? How they are obtained?

Answer: Miller indices are the number by which orientation of plane related to the crystal axis can be represented w.r.t. origin.

Process

(i) Find out intercept of the plane on crystal axis (*pa, qb, rc*).

(ii) Take reciprocal of intercept ($1/p$, $1/q$, $1/r$).

Simplest ratio are Miller indices (*h k l*). For more information see Chapter 8, Sec. 8.12.

4. Attempt any one part of the following: **(1 × 5 = 5)**

(a) What is Bragg's law? Describe Bragg's spectrometer and explain how it is used to study the crystal structure.

Answer: Bragg law, Proof of $2d\sin\theta = n\lambda$. See Chapter 8, Sec. 8.14.2.

(b) Explain briefly the different types of polarization in dielectrics.

Answer: Types of polarization: (a) electronic, (b) ionic, (c) orientation, and (d) space charge.

See Chapter 9, Secs. 9.4 and 9.5.

5. Attempt any one part of the following: **(1 × 5 = 5)**

   (a) Discuss Lorentz field equation for a non-polar isotropic dielectric.

   Answer: The total field at the atomic site is called the internal field.

$$E' = E + \frac{\gamma P}{\varepsilon_0}, \text{ for cubic structure } \gamma = 1/3$$

   See Chapter 9, Sec. 9.10.

   (b) What do you mean by hysteresis loss? Show that it is equal to the area of the hysteresis curve.

   Answer: Hysteresis loss is loss of residual magnetizing energy in the form of heat.

   Derivation $W = \int_{B1}^{B2} H dB$ (See Chapter 10, Sec. 10.7)

6. Attempt any one part of the following: **(1 × 5 = 5)**

   (a) Explain the concept of displacement current and show how it led to the modification of Ampere's law.

   Answer: The changing electric field is equivalent to a current which produces magnetic field. This equivalent current is known as displacement current.

   For Modified Ampere's law, see Chapter 11, Sec. 11.4.

   (b) What is skin depth? Show that for poor conductors, skin depth is independent of frequency of wave.

   Answer: Distance in which strength of electric field associated with EMW reduces to $1/e$ times of initial value is called skin depth. Skin depth, for poor conductors it independent of frequency.

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\in}{\mu}}$$

   For derivation to show independence of frequency, see Ch. 11, Section 11.14.

7. Attempt any one part of the following: **(1 × 5 = 5)**

   (a) Differentiate between intrinsic and extrinsic semiconductor on the basis of Fermi level.

   Answer: See Chapter 12, Sec 12.10.

   (b) What are superconductors? Explain the effect of magnetic field on superconductor.

   Answer: See Chapter 13, Secs. 13.1 and 13.5.

$$H_c = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

About the Book

This book has been written to serve as a text for revised syllabus of Engineering Physics-I (NAS-101), Engineering Physics-II (NAS-201) and Practical (NAS-151/251) of Uttar Pradesh Technical University (UPTU), Lucknow. The requirements of the students have been fulfilled by explaining the basic principles and applications of Engineering Physics in a simple, lucid and systematic manner. Extensive care has also been taken in such a manner that the beginners will also grasp and understand the complex ideas easily with appropriate examples, neatly drawn diagrams, tables wherever required, etc.

The accompanying LAB MANUAL provides detailed theory, method, observation table and question and answer for viva-voce. It provides complete information on all experiments prescribed as per UPTU syllabus.

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